# Electronic Companion for "Dynamic Scheduling of Outpatient Appointments under Patient No-shows and Cancellations" by Liu, Ziya, and Kulkarni

## Appendix A - Proofs of the Results

**Proof of Proposition 1:** To show that $f_j(y_j, \mathbf{x}, \mathbf{p})$ is concave in $y_j$, it suffices to show that $\mathbf{E}\left[r\left(w+\bar{W}_j(\mathbf{y}), v+\bar{V}_j(\mathbf{y})\right)\right]$ is concave in $y_j$ for any real numbers $w$ and $v$. Now, notice that $\bar{W}_j(\mathbf{y})$ and $\bar{V}_j(\mathbf{y})=i$ are dependent. Furthermore, conditioning on $\bar{V}_j(\mathbf{y})=i$, $\bar{W}_j(\mathbf{y})$ is a binomial random variable with parameters $i$ and $\alpha_{0j}/\beta_{0j}$. A direct application of Lemma 1 yields the result. ■.

**Lemma 1.** Define $B(i,p)$ to be a Binomial random variable with parameters $i$ and $p$. Let $\{W_j, j=0,1,2.\dots\}$ and $\{V_j, j=0,1,2.\dots\}$ be two sequences of random variables such that (1) $W_j \stackrel{d}{=} B(j,\alpha)$, (2) $V_j \stackrel{d}{=} B(j,\beta)$ and (3) conditioning on $V_j=i$, $W_j \stackrel{d}{=} B(i,\alpha/\beta)$ for $0 \le i \le j$, where $0 \le \alpha \le \beta \le 1$. Suppose that $r(x,z)$ is submodular and jointly concave in $x$ and $z$. For any arbitrary real numbers $w$ and $v$, let $g(j)=\mathbf{E}\{r(W_j+w, V_j+v)\}$. Then $g(j)$ is a concave function over the set of non-negative integers, i.e., $g(j+2)+g(j) \le 2g(j+1)$, $\forall j=0,1,2,\dots$.

**Proof:** The proof uses a coupling argument. Let $\{Z_{i,k}, k=0,1,2,\dots\}$ be a sequence of i.i.d. Bernoulli random variables with parameter $\beta$ for each $k \in \{1,2,3,4\}$. First, we write

$$
\begin{aligned}
&g(j+2)+g(j)-2g(j+1)\\
=&\mathbf{E}\{r(B(\sum_{k=1}^{j+2} Z_{1,k}, \alpha/\beta)+w, \sum_{k=1}^{j+2} Z_{1,k}+v)+r(B(\sum_{k=1}^{j} Z_{2,k}, \alpha/\beta)+w, \sum_{k=1}^{j} Z_{2,k}+v)\\
&-r(B(\sum_{k=1}^{j+1} Z_{3,k}, \alpha/\beta)+w, \sum_{k=1}^{j+1} Z_{3,k}+v)-r(B(\sum_{k=1}^{j+1} Z_{4,k}, \alpha/\beta)+w, \sum_{k=1}^{j+1} Z_{4,k}+v)\}. \quad \text{(A-1)}
\end{aligned}
$$

Now we couple the random variables so that $Z_{1,k}=Z_{2,k}=Z_{3,k}=Z_{4,k}=z_k$ for $k=0,1,2,\dots,j$, $Z_{1,j+1}=Z_{3,j+1}=\hat{z}$ and $Z_{1,j+2}=Z_{4,j+1}=\tilde{z}$. There are four possible cases for the pair $(\hat{z},\tilde{z})$: (i) $\hat{z}=\tilde{z}=0$, (ii) $\hat{z}=1$ and $\tilde{z}=0$, (iii) $\hat{z}=0$ and $\tilde{z}=1$ and (iv) $\hat{z}=\tilde{z}=1$. For case (i), (ii) and (iii), (A-1) reduces to 0. In the rest of the proof, we show that (A-1) is also non-positive under case (iv).

Let $z=\sum_{k=1}^{j} z_k$. Then, under case (iv), we have

$$
\begin{aligned}
&g(j+2)+g(j)-2g(j+1)\\
=&\mathbf{E}\{r(B(z+2,\alpha/\beta)+w, z+2+v)+r(B(z,\alpha/\beta)+w, z+v)\\
&-r(B(z+1,\alpha/\beta)+w, z+1+v)-r(B(z+1,\alpha/\beta)+w, z+1+v)\}. \quad \text{(A-2)}
\end{aligned}
$$

Following as above, we define $\{U_{i,k}, k=0,1,2,\dots\}$ be a sequence of i.i.d. Bernoulli random variables with parameter $\alpha/\beta$ for each $k \in \{1,2,3,4\}$. Then we write equation (A-2) as

$$\begin{aligned}
&g(j+2)+g(j)-2g(j+1)\\
=&\mathbf{E}\{r(\sum_{k=1}^{z+2} U_{1,k}+w, z+2+v)+r(\sum_{k=1}^{z} U_{2,k}+w, z+v)\\
&-r(\sum_{k=1}^{z+1} U_{3,k}+w, z+1+v)-r(\sum_{k=1}^{z+1} U_{4,k}+w, z+1+v)\}.
\end{aligned} \tag{A-3}$$

Now we couple the random variables so that $U_{1,k}=U_{2,k}=U_{3,k}=U_{4,k}=u_k$ for $k=0,1,2,\ldots,z$, $U_{1,z+1}=U_{3,z+1}=\hat{u}$ and $U_{1,z+2}=U_{4,z+1}=\tilde{u}$. There are four possible cases for the pair $(\hat{u},\tilde{u})$:

Case 1: $\hat{u}=\tilde{u}=0$. Then the term inside the expectation in (A-3) becomes

$$r(\sum_{k=1}^{z} u_k+w, z+2+v)+r(\sum_{k=1}^{z} u_k+w, z+v)-r(\sum_{k=1}^{z} u_k+w, z+1+v)-r(\sum_{k=1}^{z} u_k+w, z+1+v)\le 0$$

due to the concavity of $r(\cdot,\cdot)$.

Case 2: $\hat{u}=1$ and $\tilde{u}=0$. Then the term inside the expectation in (A-3) becomes

$$r(\sum_{k=1}^{z} u_k+1+w, z+2+v)+r(\sum_{k=1}^{z} u_k+w, z+v)-r(\sum_{k=1}^{z} u_k+1+w, z+1+v)-r(\sum_{k=1}^{z} u_k+w, z+1+v)\le 0$$

since

$$\begin{aligned}
&r(\sum_{k=1}^{z} u_k+1+w, z+2+v)-r(\sum_{k=1}^{z} u_k+1+w, z+1+v)\\
\le& r(\sum_{k=1}^{z} u_k+1+w, z+1+v)-r(\sum_{k=1}^{z} u_k+1+w, z+v) \quad \text{(due to the concavity of } r(\cdot))\\
\le& r(\sum_{k=1}^{z} u_k+w, z+1+v)-r(\sum_{k=1}^{z} u_k+w, z+v). \quad \text{(due to the submodularity of } r(\cdot,\cdot))
\end{aligned}$$

Case 3: $\hat{u}=0$ and $\tilde{u}=1$. Then the term inside the expectation in (A-3) becomes

$$r(\sum_{k=1}^{z} u_k+1+w, z+2+v)+r(\sum_{k=1}^{z} u_k+w, z+v)-r(\sum_{k=1}^{z} u_k+w, z+1+v)-r(\sum_{k=1}^{z} u_k+1+w, z+1+v)\le 0,$$

following the same proof in Case (2) above.

Case 4: $\hat{u}=\tilde{u}=1$. Then the term inside the expectation in (A-3) becomes

$$r(\sum_{k=1}^{z} u_k+2+w, z+2+v)+r(\sum_{k=1}^{z} u_k+w, z+v)-r(\sum_{k=1}^{z} u_k+1+w, z+1+v)-r(\sum_{k=1}^{z} u_k+1+w, z+1+v)\le 0$$

due to the concavity of $r(\cdot,\cdot)$. Hence, the result follows.■

**Proof of Proposition 2:** Let $\lambda$ denote the mean number of daily arrivals and $PO(\alpha)$ denote a Poisson random variable with mean $\alpha$. Then,

$$\tilde{V}_0(p_0) \stackrel{d}{=} B(\tilde{A}_0, p_0) \stackrel{d}{=} PO(\lambda p_0), \tilde{V}_1(p_0) \stackrel{d}{=} B(\tilde{A}_1, (1-p_0)\beta_{01}) \stackrel{d}{=} PO(\lambda(1-p_0)\beta_{01}),$$

and

$$\tilde{W}_0(p_0) \stackrel{d}{=} B(\tilde{A}_0, p_0\alpha_{00}) \stackrel{d}{=} PO(\lambda p_0 \alpha_{00}), \tilde{W}_1(p_0) \stackrel{d}{=} B(\tilde{A}_1, (1-p_0)\alpha_{01}) \stackrel{d}{=} PO(\lambda(1-p_0)\alpha_{01}).$$

Note that $\tilde{V}_0(p_0)$ is independent of $\tilde{V}_1(p_0)$ and $\tilde{W}_0(p_0)$ is independent of $\tilde{W}_1(p_0)$ since $\tilde{A}_0$ is independent of $\tilde{A}_1$. Let $C_1(p_0)$ be a random variable such that $C_1(p_0) \stackrel{d}{=} PO(\lambda p_0(\alpha_{00} - \alpha_{01}))$ (note that $\alpha_{00} - \alpha_{01} \geq 0$), and $D_1 \stackrel{d}{=} PO(\lambda \alpha_{01})$ be a random variable which is independent of $C_1(p_0)$. Also, let $C_2(p_0)$ be a random variable such that $C_2(p_0) \stackrel{d}{=} PO(\lambda p_0(1-\beta_{01}))$, and $D_2 \stackrel{d}{=} PO(\lambda\beta_{01})$ be a random variable which is independent of $C_2(p_0)$. Then, using the fact that the sum of two independent Poisson random variables is another Poisson random variable, we have

$$\tilde{W}_0(p_0) + \tilde{W}_1(p_0) \stackrel{d}{=} PO(\lambda p_0(\alpha_{00} - \alpha_{01}) + \lambda\alpha_{01}) \stackrel{d}{=} C_1(p_0) + D_1$$

and

$$\tilde{V}_0(p_0) + \tilde{V}_1(p_0) \stackrel{d}{=} PO(\lambda p_0(1-\beta_{01}) + \lambda\beta_{01}) \stackrel{d}{=} C_2(p_0) + D_2.$$

Assumption 1 implies that $w(\cdot)$ is an increasing convex function and $r(\cdot)$ is an increasing concave function. Let $z^+ = \max\{z, 0\}$. Since the class of functions $g_s(z) = (z-s)^+$ for all $s \in \mathbb{R}$ generates all the increasing convex functions and the class of functions $h_s(z) = -(s-z)^+$ for all $s \in \mathbb{R}$ generates all the increasing concave functions (see Shaked and Shanthikumar 1994), in order to show

$$\mathbf{E}\left[r\left(\tilde{W}_0(p_0) + \tilde{W}_1(p_0)\right) - w\left(\tilde{V}_0(p_0) + \tilde{V}_1(p_0)\right)\right]$$

is concave in $p_0$ it suffices to show that

$$\mathbf{E}\{-[s - (C_1(p_0) + D_1)]^+\}$$

is concave in $p_0$ for all $s \in \mathbb{R}$ and

$$\mathbf{E}\{[(C_2(p_0) + D_2) - s]^+\}$$

is convex in $p_0$ for all $s \in \mathbb{R}$, both of which immediately follow from Lemma 2 shown and proved below. ■

**Lemma 2.** Let $z^+ = \max\{z, 0\}$. Suppose that $\tilde{C}(p)$ is a Poisson random variable with mean $p\theta$ where $p, \theta > 0$ and $A$ is a random variable that is independent of $\tilde{C}(p)$. Then $\mathbf{E}\{[(\tilde{C}(p) + A) - s]^+\}$ and $\mathbf{E}\{[s - (\tilde{C}(p) + A)]^+\}$ are both convex in $p$ for any given $s \in \mathbb{R}$.

**Proof:** It is sufficient to show that for all $a \in \mathbb{R}$, $\mathbf{E}\{[(\tilde{C}(p) + A) - s]^+|A = a\}$ and $\mathbf{E}\{[s - (\tilde{C}(p) + A)]^+|A = a\}$ are both convex in $p$ for an arbitrary $s \in \mathbb{R}$. We first show that $\mathbf{E}\{[(\tilde{C}(p) + A) - s]^+|A = a\}$ is convex in $p$ for any given $a, s \in \mathbb{R}$. This is equivalent to showing that $\mathbf{E}\{[\tilde{C}(p) - s]^+\}$ is convex in $p$ for any given $s \in \mathbb{R}$.

Now, to show that $\mathbf{E}\{[\tilde{C}(p) - s]^+\}$ is convex in $p$, first note that the proof is trivial if $s \leq 0$, and thus we only need to consider $s > 0$. Denote $\lceil s \rceil$ to be the smallest integer that is larger than or equal to $s$, and let $g(\cdot|\lambda)$ be the probability mass function of a Poisson random variable with mean $\lambda$, i.e.

$$g(i|\lambda) = \begin{cases} e^{-\lambda}\frac{\lambda^i}{i!}, & \text{if } i = 0, 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

After some straightforward but tedious calculus, one can show that for any $s > 0$,

$$\frac{d^2}{dp^2}\mathbf{E}\{[\tilde{C}(p) - s]^+\} = \theta^2[(s + 1 - \lceil s \rceil)g(\lceil s \rceil - 1|\theta p) + (\lceil s \rceil - s)g(\lceil s \rceil - 2|\theta p)] \geq 0,$$

which establishes that $\mathbf{E}\{[\tilde{C}(p) - s]^+\}$ is convex in $p$ for any given $s \in \mathbb{R}$.

Similarly, to show $\mathbf{E}\{[s-(\tilde{C}(p)+A)]^+\}$ is convex in $p$ for any given $s \in \mathbb{R}$, it is sufficient to show that $\mathbf{E}\{[s-\tilde{C}(p)]^+\}$ is convex in $p$ for any given $s \in \mathbb{R}$. But then since $\mathbf{E}\{[s-\tilde{C}(p)]^+\} = \mathbf{E}\{[\tilde{C}(p) - s]^+ + [s - \tilde{C}(p)]\} = \mathbf{E}\{[\tilde{C}(p] - s)^+\} + s - \theta p$, the result immediately follows. ■.

**Proof of Theorem 1:** Let $f^*$ denote the policy that schedules all appointment requests received today for the $j^*$th day from today where $j^* = \arg\max_{j \in 0,1,\dots,T} I_j$ where $I_j$ is as described in (12).

First, the long-run average net reward under policy $f^*$ is $\phi_{f^*} = (\tau\alpha_{0j^*} - \nu_1\beta_{0j^*})\mu$. For almost all sample paths $w$ of $\{A^1, A^2, A^3, \dots\}$, we show that, for any arbitrary policy $f$, the long-run average net reward along this path, denoted by $\phi_f(\omega)$ , is no larger than $\phi_{f^*}$. Without loss of generality, we assume that the initial backlog is empty, i.e. $\mathbf{X}^0 = \mathbf{0}$. Let $N_j(t, \omega)$ be the total number of patients scheduled with appointment delay $j$ days up to day $t$ along sample path $\omega$ under policy $f$. Now, since the cost and reward are both linear, we can view the total net reward as the sum of net rewards contributed by individual patients. Let $R_{ij}(t, \omega)$ be the net reward generated by the $i$th patient of those $N_j(t, \omega)$ patients whose appointment delay is $j$ days along sample path $\omega$ up to day $t$, $i = 1, 2, \dots, N_j(t, \omega)$. Notice that $\{R_{ij}(t, \omega), i = 1, 2, \dots, N_j(t, \omega)\}$ is the realization of a sequence of i.i.d. random variables with mean $\tau\alpha_{0j} - \nu_1\beta_{0j}$ along sample path $\omega$. Let

$\mathcal{J} = \{j : \lim_{t\to\infty} N_j(t,\omega) = \infty\}$. Then,

$$\begin{aligned}
\phi_f(\omega) &= \lim_{t\to\infty} \frac{\sum_{j=0}^{T}\sum_{i=1}^{N_j(t,\omega)} R_{ij}(t,\omega)}{t} \\
&= \lim_{t\to\infty} \frac{1}{t}\sum_{j=0}^{T} N_j(t,\omega)\frac{\sum_{i=1}^{N_j(t,\omega)} R_{ij}(t,\omega)}{N_j(t,\omega)} \\
&= \lim_{t\to\infty} \frac{1}{t}\sum_{j\in\mathcal{J}} N_j(t,\omega)\frac{\sum_{i=1}^{N_j(t,\omega)} R_{ij}(t,\omega)}{N_j(t,\omega)} \\
&= \lim_{t\to\infty} \frac{1}{t}\sum_{j\in\mathcal{J}} N_j(t,\omega)\lim_{t\to\infty}\frac{\sum_{i=1}^{N_j(t,\omega)} R_{ij}(t,\omega)}{N_j(t,\omega)} \\
&= \lim_{t\to\infty} \frac{1}{t}\sum_{j\in\mathcal{J}} N_j(t,\omega)(\tau\alpha_{0j} - \nu_1\beta_{0j}) \quad \text{(by Strong Law of Large Numbers)} \\
&\leq (\tau\alpha_{0j^*} - \nu_1\beta_{0j^*})\lim_{t\to\infty}\frac{1}{t}\sum_{j\in\mathcal{J}} N_j(t,\omega) \quad \text{(by the definition of } j^*) \\
&\leq (\tau\alpha_{0j^*} - \nu_1\beta_{0j^*})\mu \quad (\text{since } \lim_{t\to\infty}\frac{1}{t}\sum_{j\in\mathcal{J}} N_j(t,\omega) \leq \lim_{t\to\infty}\frac{1}{t}\sum_{j=0}^{T} N_j(t,\omega) = \mu) \\
&= \phi_{f^*}.
\end{aligned}$$

This completes the proof. ■.

# Appendix B - Derivations of MLEs for $\alpha_{ij}$ and $\beta_{ij}$

From the data obtained from the Department of Family Medicine at the University of North Carolina, we extracted the following information:
$C_i$: Number of patients who had an appointment delay of $i$ days but canceled their appointments on or before their appointment days.
$S_i$: Number of patients who had an appointment delay of $i$ days and showed up for their appointments.
$M_i$: Number of patients who had an appointment delay of $i$ days, did not cancel in advance but missed their appointments.

First, define the following:

$$\begin{aligned}
q_i &= \mathbf{P}(\text{NS}|T_c \geq i+1), \\
r_i &= \mathbf{P}(\text{S}|T_c \geq i+1), \\
u_i &= \mathbf{P}(T_c \leq i).
\end{aligned} \tag{A-4}$$

Note that $q_i$ and $r_i$ are respectively the probabilities of the "patient no-show" and "patient show" events given that the patient does not cancel in the first $i+1$ days after the day she calls for an appointment; $u_i$ is the probability that a patient will cancel no later than $i$ days after she calls for an appointment.

Clearly, we must have $q_i + r_i = 1$ and $u_i \le u_{i+1}$, $i \in \{0, 1, \ldots, T\}$. Recall that each patient's cancellation and no-show behaviors are independent of those of other patients and for any appointment made, there are three possible outcomes: the patient cancels any time on or before the appointment day, the patient misses the appointment without cancellation, and the patient shows up for the appointment. Let $\mathbf{q} = \{q_i\}_{i=0}^T$, $\mathbf{r} = \{r_i\}_{i=0}^T$ $\mathbf{u} = \{u_i\}_{i=0}^T$. Then the MLEs for $q_i$, $r_i$, and $u_i$ can be obtained by solving the following maximization problem

$$\begin{array}{ll} \max & L(\mathbf{q}, \mathbf{r}, \mathbf{u}) = \prod_{i=0}^T u_i^{C_i} [(1-u_i) q_i]^{M_i} [(1-u_i) r_i]^{S_i} \\ \text{s.t.} & q_i + r_i = 1, \quad i \in \{0, 1, \ldots, T\}, \\ & u_i \le u_{i+1}, \quad i \in \{0, 1, \ldots, T-1\}, \\ & u_i \ge 0, q_i \ge 0, r_i \ge 0, \quad i \in \{0, 1, \ldots, T\}. \end{array} \tag{A-5}$$

Suppose that we solve the optimization problem (A-5) and obtain the MLEs $\hat{q}_i$, $\hat{r}_i$, and $\hat{u}_i$ for $q_i$, $r_i$, and $u_i$, respectively. Then, we can get the MLEs $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ for $\alpha_{ij}$ and $\beta_{ij}$ as follows:

$$\begin{array}{rcl} \hat{\alpha}_{ij} & = & \frac{\hat{r}_{i+j}(1-\hat{u}_{i+j})}{1-\hat{u}_{i-1}}, \\ \hat{\beta}_{ij} & = & \frac{1-\hat{u}_{i+j-1}}{1-\hat{u}_{i-1}}. \end{array} \tag{A-6}$$

where $\hat{u}_{-1} = 0$ by definition.

However, solving Problem (A-5) is difficult especially since it has too many decision variables, i.e., there are too many parameters to estimate. Hence, we propose a parsimonious parametric model, which requires estimating only four parameters. Specifically, assume that $q_i$, $r_i$, and $u_i$ take the following form:

$$\begin{array}{rcl} q_i & = & 1 - \theta b^{i+1}, \quad i \ge -1, \\ r_i & = & \theta b^{i+1}, \quad i \ge -1, \\ u_i & = & \begin{cases} 0, & i = -1, \\ 1 - \gamma a^i, & i \ge 0. \end{cases} \end{array} \tag{A-7}$$

This model is appealing not only because it is sufficiently simple but also because it has an interpretation that is quite fitting in the appointment scheduling context. First, note that $u_i$ is the cumulative distribution function for the random variable $T_c$, i.e. the time between the patient's call and the day she cancels (or the day she would cancel if her appointment was not earlier). Under the parametric form we describe in (A-7), the probability mass function of $T_c$ is a mixture of two distributions, one being a constant and the other being a geometric distribution. To be more precise, we have

$$\mathbf{P}(T_c = i) = (1-\gamma)\mathbf{1}_{\{i=0\}} + \gamma \mathbf{P}(Y_c = i)\mathbf{1}_{\{i \ge 1\}},$$

where $\mathbf{1}_A$ is the indicator function and $Y_c$ is a geometric random variable with parameter $1-a$. One way of interpreting this mixture structure is that there are two different types of patients: those who cancel on the same day they make their appointments, which constitute $1-\gamma$ fraction of the whole patient population, and those who cancel later, which constitute $\gamma$ fraction of the whole patient population. Furthermore, the model also implies that for those who make appointments at least one day before their appointment day, the probability of cancelling on each day is $1-a$ independently of everything else. (Note that it is possible to make similar interpretations for $q_i$ and $r_i$ as well.)

Notice that once the probabilities are restricted to be in the form given above, the only additional condition needed for all the constraints of Problem (A-5) to hold is that $0 \leq \gamma, a, \theta, b \leq 1$. Let $\hat{\gamma}, \hat{a}, \hat{\theta}$, and $\hat{b}$ denote the MLEs for $\gamma, a, \theta$, and $b$, respectively. Then, the first order optimality condition yields

$$\begin{aligned}
\sum_{i=0}^{T}(C_i + M_i + S_i) &= \sum_{i=0}^{T} \frac{C_i}{1 - \hat{\gamma}\hat{a}^i}, \\
\sum_{i=0}^{T} i(C_i + M_i + S_i) &= \sum_{i=0}^{T} \frac{iC_i}{1 - \hat{\gamma}\hat{a}^i}, \\
\sum_{i=0}^{T}(M_i + S_i) &= \sum_{i=0}^{T} \frac{M_i}{1 - \hat{\theta}\hat{b}^{i+1}}, \\
\sum_{i=0}^{T}(i+1)(M_i + S_i) &= \sum_{i=0}^{T} \frac{(i+1)M_i}{1 - \hat{\theta}\hat{b}^{i+1}}.
\end{aligned}$$

This system of nonlinear equations can be solved by one of the standard algorithms such as Gauss-Newton method or Trust-Region method. Once this solution is found, the estimates for $\alpha_{ij}$ and $\beta_{ij}$ can be determined using the following equations, which are obtained by simply substituting (A-7) into (A-6):

$$\hat{\alpha}_{ij} = \begin{cases} \hat{\theta}\hat{b}^{j+1}\hat{\gamma}\hat{a}^j & \text{if } i = 0, \\ \hat{\theta}\hat{b}^{i+j+1}a^{j+1} & \text{if } i \geq 1, \end{cases}$$

$$\hat{\beta}_{ij} = \begin{cases} 1 & \text{if } i = 0, j = 0, \\ \hat{\gamma}\hat{a}^{j-1} & \text{if } i = 0, j \geq 1, \\ \hat{a}^j & \text{if } i \geq 1, j \geq 1. \end{cases}$$

Using our data, we found that $\hat{\gamma} = 0.9297$, $\hat{a} = 0.9987$, $\hat{\theta} = 0.8863$, and $\hat{b} = 0.9953$. Then, we numerically verified that this solution is indeed a maximizer. In order to build confidence for our statistical model, we have also conducted a Chi-square goodness-of-fit test, and we found that the distribution we proposed can not be rejected at the significance level of 0.01 (the p-value is 0.042).