TECHNICAL NOTE

Relationships Among Three Assumptions in Revenue Management

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This note discusses the relationships among three assumptions that appear frequently in the pricing/revenue management literature. These assumptions are mostly needed for analytical tractability, and they have the common property of ensuring a well-behaved "revenue function." The three assumptions are decreasing marginal revenue with respect to demand, decreasing marginal revenue with respect to price, and increasing price elasticity of demand. We provide proofs and examples to show that none of these conditions implies any other. However, they can be ordered from strongest to weakest over restricted regions, and the ordering depends upon the region.

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1. Introduction

Our aim is to discuss the relationships among three assumptions that are prevalent in the pricing/revenue management literature. Our motivation, and that of many other researchers, comes from a pricing problem where the mathematical relationship between price and demand is known, and revenue can be defined as a function of either price or demand. The assumptions we investigate in this note help researchers to have analytically tractable models; among others, they have the common property of ensuring that the "revenue function" is either unimodal or monotone (i.e., quasi-concave). Although our primary objective is to investigate whether any of these assumptions is more or less restrictive than any of the other two, our results also have economic implications because the assumptions have economic interpretations. In the following sections, we first define the model and state the assumptions of interest. Then, we show that these assumptions imply each other over restricted regions of price and demand. Finally, we provide counterexamples that demonstrate that, in general, none of these assumptions is more restrictive than any other.

2. The Model and the Three Assumptions

We consider a fairly standard price-demand formulation, which assumes that the mathematical relationship between

price and demand is known. We use x to denote the price (of a product, service, etc.) and y to denote the demand (demand in a time period, demand per time, etc.). We use $d(x)$ to denote the demand corresponding to price x. We assume that $d(\cdot)$ is a nonincreasing, continuous function on $[0, \infty)$. Initially, we assume that $d(\cdot)$ is bounded, and $d(\infty) = 0$ (where $d(\infty) = \lim_{x\to\infty} d(x)$), but we relax these assumptions in §7. Let $D = d(0)$ denote the least upper bound on the demand with the condition $0 < D < \infty$. We further assume that $d(\cdot)$ is strictly decreasing and twice differentiable over $(x_{\min}, x_{\max})$, where $x_{\min} = \sup\{x \geqslant 0: d(x) = D\}$ and $x_{\max} = \inf\{x \geqslant 0: d(x) = 0\}$. Throughout the paper, we adopt the convention that $\inf \varnothing = \infty$. Then, $0 \leqslant x_{\min} < x_{\max} \leqslant \infty$. Note that $d(x_{\min}) = D$ and $d(x_{\max}) = 0$ because $d(\cdot)$ is a continuous function.

Because the function $d(\cdot)$ is bounded, it can be rewritten as

$$d(x) = D(1 - F(x)) \quad \text{for } x \in [0, \infty), \tag{1}$$

where $0 \leqslant F(x) \leqslant 1$ for $x \geqslant 0$. One way to interpret D and the function $F(\cdot)$ is as follows. The parameter D is the potential number of customers (or the potential arrival rate of customers), that is, the demand the manager faces when the price is set to zero. If the price is set to x, each potential customer decides to buy the product (or join the system) with a probability of $1 - F(x)$. Then, $d(x)$ is the

expected demand or the demand rate given price x, and $F(\cdot)$ is a cumulative distribution function representing customers' willingness-to-pay. We also define function $f(\cdot)$ as the p.d.f. and $h(\cdot)$ as the hazard rate function for $F(\cdot)$ $(h(x) = f(x)/(1 - F(x)))$.

Because $d(\cdot)$ is strictly decreasing over $(x_{\min}, x_{\max})$, there is a unique price corresponding to each demand value $y \in (0, D)$, and the function $d(\cdot)$ has an inverse over $(x_{\min}, x_{\max})$. We define the function $p(\cdot)$ as the inverse demand function. Given the demand value, $p(\cdot)$ returns the corresponding price. Then, using (1), we can define the function $p(\cdot)$ as

$$p(y) = F^{-1}\left(1 - \frac{y}{D}\right) \quad \text{for } y \in (0, D), \tag{2}$$

where $F^{-1}(\cdot)$ is the inverse of $F(\cdot)$. Without loss of generality, we let $p(0) = x_{\max}$ and $p(D) = x_{\min}$.

We now have the flexibility of defining what we call the "revenue function" either in terms of price or in terms of demand. We use $R_p(\cdot)$ to denote the revenue as a function of price, which can be written as

$$R_p(x) = xd(x) \quad \text{for } x \in [0, \infty). \tag{3}$$

Similarly, we use $R_d(\cdot)$ to denote the revenue as a function of demand:

$$R_d(y) = yp(y) \quad \text{for } y \in [0, D]. \tag{4}$$

Obviously, we have $R_p(x) = R_d(d(x))$ for $x \in [x_{\min}, x_{\max}]$ and $R_p(p(y)) = R_d(y)$ for $y \in [0, D]$.

In the literature, there is a variety of work where the revenue function in the form of (3) or (4) comes up. Different authors make different assumptions on these functions (or on functions $d(\cdot)$ and $p(\cdot)$) to have analytically tractable models. The purpose of this note is to explore the relationships among the following three assumptions: (Note that, in a sense, there are six assumptions because each assumption has two variations, "strict" and "nonstrict.")

ASSUMPTION A1. (STRICT) CONCAVITY IN DEMAND. $R_d(y)$ is (strictly) concave for $y \in (0, D)$.

ASSUMPTION A2. (STRICT) CONCAVITY IN PRICE. $R_p(x)$ is (strictly) concave for $x \in (x_{\min}, x_{\max})$.

ASSUMPTION A3. $F(\cdot)$ HAS A (STRICTLY) INCREASING GENERALIZED FAILURE RATE—IGFR. $e(x) = xh(x)$ is (strictly) increasing for $x \in (x_{\min}, x_{\max})$.

We adopt the terminology of Lariviere and Porteus (2001) referring to function $e(\cdot)$ as the "generalized failure rate" and use their definition of "IGFR."

One common property of these assumptions is that each ensures a well-behaved revenue function. To be more precise, under any of these assumptions, it can be shown that $R_p(\cdot)$ (or $R_d(\cdot)$) is either (strictly) unimodal or monotone. Note that, from our model assumptions, it follows

that $R_p(x)$ is strictly increasing for $0 \leqslant x < x_{\min}$ while $R_p(x) = 0$ for $x \geqslant x_{\max}$.

A1 and A2 both require the revenue function to be concave, the former in demand, the latter in price. Hence, A1 and A2 can be regarded as duals to each other. As we will demonstrate in the following sections, these are two different conditions. To construct a dual to A3, assume that price is bounded so that $p(\cdot)$ can be expressed as $p(y) = P(1 - H(y))$ for some distribution function $H(\cdot)$ with $P < \infty$. Then, a dual condition to A3 can be stated as:

ASSUMPTION A4. $H(\cdot)$ *has an increasing generalized failure rate.*

However, there is no reason to include both A3 and A4 due to the following result (see the appendix for the proof):

PROPOSITION 2.1. *Assumptions* A3 *and* A4 *are equivalent.*

Assumptions A1, A2, and A3 also have some economic implications. A1 implies that marginal revenue with respect to demand is decreasing, whereas A2 implies that marginal revenue with respect to price is decreasing. It can be shown that $e(\cdot)$ is the price elasticity of the demand function, which can be defined as

$$e(x) = -\lim_{\Delta x \to 0} \frac{(d(x + \Delta x) - d(x))/d(x)}{\Delta x/x}.$$

(Also, see Lariviere and Porteus 2001 for an earlier work that relates price elasticity to the hazard rate with a different model formulation.) Thus, A3 is equivalent to increasing price elasticity.

We are interested in whether any of these three assumptions is weaker or stronger than any of the other(s). Lariviere and Porteus (2001) point out that A2 fails for most common distributions, such as the normal distribution, while A3 holds for most common distributions (e.g., normal, uniform, gamma). In the following sections, we will show that none of these assumptions is more restrictive than any other. However, we will prove some implications among these assumptions over restricted regions of price and demand.

3. Papers Using These Assumptions

To our knowledge, A1 is the most widely used of the three assumptions stated in §2. For example, see Feichtinger and Hartl (1985), Li (1988), Gallego and van Ryzin (1994), Paschalidis and Tsitsiklis (2000), and Chatwin (2000) (with a finite set of prices). Bitran and Mondschein (1997) assume that the function $(1 - F_t(x))^2/f_t(x)$ (where subscript t is for time) is decreasing in x, and Cachon and Lariviere (2001) assume that the function $1/(1 - F(x))$ is convex. It can be shown that both of these assumptions are equivalent to A1 if $F_t(x)$ and $F(x)$ are twice differentiable in x. Note that in Cachon and Lariviere (2001), $F(\cdot)$ has a different meaning; it is the probability distribution function

for the demand. A common assumption in the auction and mechanism design literature is that the function $x - 1/h(x)$ is increasing, which is also equivalent to A1 (e.g., see McAfee and McMillan 1987 and Bulow and Roberts 1989). In general, it seems that researchers find it more convenient to make the demand decision variable (rather than the price), and therefore assuming concavity in price (A2) is not common. For an example, see Yoshida (2002).

On the other hand, Lariviere and Porteus (2001), Fridgeirsdottir and Chiu (2001), and Ziya et al. (2003) assume A3. Because Lariviere and Porteus (2001) work in quantities rather than prices, in their paper D and $F(\cdot)$ have completely different meanings. Function $F(\cdot)$ is the probability distribution for the demand. However, the form of their revenue function is exactly the same as (3) with the function $d(\cdot)$ defined as in (1) (although it is not a demand function), and therefore their assumption is technically the same as A3. In fact, even if Lariviere and Porteus had worked in prices, their assumption could have been interpreted as an A4 assumption, which has been shown to be equivalent to A3. Fridgeirsdottir and Chiu (2001) also consider the assumption that the function $h(x)$ is strictly increasing. This assumption obviously implies A3, but the authors show that it also implies A1. Hempenius (1970) considers a profit function that involves a cost term and gives sufficient conditions that ensure that the solution to the first-order condition is the optimal solution. Hempenius shows that in addition to the convexity assumption for the cost, it is sufficient to assume either A3 or another condition, which is equivalent to A1.

4. Equivalent Conditions

Assumptions A1, A2, and A3 are difficult to compare because they are quite different on the surface. In this section, we give three conditions—C1, C2, and C3—that are equivalent to A1, A2, and A3, respectively. The advantage of C1, C2, and C3 is that they are all expressed in terms of the functions $f(\cdot)$, $f'(\cdot)$, and $h(\cdot)$.

These conditions are:

CONDITION C1. $2h(x) \geqslant (>) -f'(x)/f(x)$ for $x \in (x_{\min}, x_{\max})$.

CONDITION C2. $2/x \geqslant (>) -f'(x)/f(x)$ for $x \in (x_{\min}, x_{\max})$.

CONDITION C3. $1/x + h(x) \geqslant (>) -f'(x)/f(x)$ for $x \in (x_{\min}, x_{\max})$.

PROPOSITION 4.1. *Assumptions* A1, A2, *and* A3 *are equivalent to Conditions* C1, C2, *and* C3, *respectively.*

Note that C1, C2, and C3 with strict inequalities $(>)$ are equivalent to the corresponding "strict" versions of A1, A2, and A3. The proof of Proposition 4.1 follows immediately from Lemma A.1 in the appendix.

5. Implications Over Restricted Regions

Even though we will show that none of Assumptions A1, A2, or A3 is more restrictive than any of the other two, there are some implications among the assumptions over restricted regions of price and demand. We first define these parameter regions.

Let x^* be defined as

$$x^* = \inf\{x : e(x) \geqslant 1\}. \tag{5}$$

If at least one of Assumptions A1, A2, or A3 holds strictly and $x^* < \infty$, then it can be shown that x^* is the unique optimal price that maximizes $R_p(x)$ and $R_p(x)$ is increasing for any $x < x^*$ and decreasing for $x_{\max} > x > x^*$. Let the sets A_p and A_d be defined as $A_p = \{x : x \in (x_{\min}, x^*]\}$ and $A_d = \{y : p(y) \in (x_{\min}, x^*]\}$. Then, A_p is the set of prices for which $R_p(x)$ is nondecreasing and A_d is the set of demand values corresponding to the prices for which $R_p(x)$ is nondecreasing. Note that A_d can also be described as the set of demand values for which $R_d(y)$ is nonincreasing. Also, let A_p^c and A_d^c denote the relative complementary sets with respect to the intervals $(x_{\min}, x_{\max})$ and $(0, D)$, respectively. Then, in the appendix, we prove the following result:

PROPOSITION 5.1. (i) *If $R_d(y)$ is (strictly) concave for $y \in A_d$, then $e(x)$ is (strictly) increasing for $x \in A_p$.*

(ii) *If $e(x)$ is (strictly) increasing for $x \in A_p$, then $R_p(x)$ is (strictly) concave for $x \in A_p$.*

(iii) *If $R_p(x)$ is (strictly) concave for $x \in A_p^c$, then $e(x)$ is (strictly) increasing for $x \in A_p^c$.*

(iv) *If $e(x)$ is (strictly) increasing for $x \in A_p^c$, then $R_d(y)$ is (strictly) concave for $y \in A_d^c$.*

Note that the fact that A3 implies A2 over the parameter region A_p (part (ii)) has already been shown by Lariviere and Porteus (2001).

Proposition 5.1 is not sufficient to claim that any of these three assumptions is stronger or weaker than the others. In fact, the nature of their relationships suggests that they are different conditions. We provide some examples, showing that this is indeed the case.

6. Counterexamples

In this section, we give three examples to show that none of Assumptions A1, A2, and A3 is more restrictive than any other. Each example satisfies only one of A1, A2, and A3. The examples also disprove the converse of the statements given in Proposition 5.1.

The following example satisfies A1, but neither A2 nor A3. Furthermore, this example shows that the converse of Proposition 5.1(iv) does not hold.

EXAMPLE 1. Suppose that $F(x) = 1 - (x - \varepsilon)^{-2}$, where $x \in [1 + \varepsilon, \infty)$ and $0 < \varepsilon < \infty$. Then, we have $h(x) = 2(x - \varepsilon)^{-1}$ and $-f'(x)/f(x) = 3(x - \varepsilon)^{-1}$ for $x \geqslant 1 + \varepsilon$.

Thus, $2h(x) > -f'(x)/f(x)$ for $x \geqslant 1+\varepsilon$, and we conclude that A1 holds. However, $2/x < -f'(x)/f(x)$ for $x \geqslant 1+\varepsilon$, which implies that A2 does not hold (in fact, $R_p(x)$ is convex for $x \geqslant 1+\varepsilon$). Finally, we have

$$e(x) = \frac{2x}{x-\varepsilon},$$

which is a decreasing function for $x \geqslant 1+\varepsilon$. Therefore, A3 does not hold. Also, because $x^* = 1+\varepsilon$, we conclude that the converse of Proposition 5.1(iv) does not hold.

The following example satisfies A2, but neither A1 nor A3. It also shows that the converse of Proposition 5.1(ii) does not hold.

EXAMPLE 2. Suppose that

$$F(x) = \begin{cases} 0.2x - x^3/3 & \text{for } 0 \leqslant x \leqslant 0.3, \\ 4x^3/3 - 3x^2/2 + 0.65x - 0.045 & \\ & \text{for } 0.3 < x \leqslant 1.241049. \end{cases}$$

Note that $x_{\max} \approx 1.241049$. Then, we have

$$-\frac{f'(x)}{f(x)} = \begin{cases} \dfrac{2x}{0.2 - x^2} & \text{for } 0 \leqslant x \leqslant 0.3, \\ \dfrac{-8x+3}{4x^2 - 3x + 0.65} & \text{for } 0.3 < x \leqslant 1.241049 \end{cases}$$

and

$$h(x) = \begin{cases} \dfrac{0.2 - x^2}{1 - 0.2x + x^3/3} & \text{for } 0 \leqslant x \leqslant 0.3, \\ \dfrac{4x^2 - 3x + 0.65}{1.0045 - 4x^3/3 + 3x^2/2 - 0.65x} & \\ & \text{for } 0.3 < x \leqslant 1.241049. \end{cases}$$

It can be shown that $2/x > -f'(x)/f(x)$ for $0 \leqslant x \leqslant 1.241049$. Thus, A2 holds. However, for $x = 0.1$, $2h(x) + f'(x)/f(x)$ is approximately -0.665008. Therefore, A1 does not hold. Finally, for $x = 0.29$, $1/x + h(x) + f'(x)/f(x)$ is approximately -1.434055. Thus, A3 does not hold. Because it can also be shown that $x^* > 0.3$, we conclude that the converse of Proposition 5.1(ii) does not hold.

Our last example below satisfies A3, but neither A1 nor A2. It also shows that the converses of Proposition 5.1(i) and (iii) do not hold. Note that an example that satisfies A3 but not A2 has also been given in Hempenius (1970).

EXAMPLE 3. Let $0 < \alpha < 1$ and $\beta > 0$, and suppose that $F \sim \text{Weibull}(\alpha, \beta)$. Then, it can be shown that $e(x) = \alpha(x/\beta)^\alpha$. Obviously, $e(x)$ is increasing so that A3 holds. We have $h(x) = (\alpha/\beta^\alpha)x^{\alpha-1}$ and $-f'(x)/f(x) = (\alpha\beta^{-\alpha}x^\alpha - (\alpha-1))/x$. Then, $2h(x) + f'(x)/f(x) < 0$ for $x \in (0, \beta((1-\alpha)/\alpha)^{1/\alpha})$. Hence, A1 does not hold. It also implies that the converse of Proposition 5.1(i) does not hold. Moreover, we have $2/x + f'(x)/f(x) < 0$ for $x \in (\beta((1+\alpha)/\alpha)^{1/\alpha}, \infty)$. This implies that A2 does not hold. Because $x^* = \beta(\alpha)^{(-1/\alpha)}$, we also conclude that the converse of Proposition 5.1(iii) does not hold. (Note that for $\alpha \geqslant 1$, it can be shown that A1 holds.)

7. Unbounded and/or Nonvanishing Demand

So far, we have assumed that demand is bounded. This allowed us to write (1) and to state A3 as an IGFR condition, which is common in the pricing/revenue management literature. To simplify the presentation, we have also assumed that demand vanishes as price goes to infinity. However, neither of these assumptions is necessary for our main result, Proposition 5.1.

We now assume that $d(\cdot)$ is an extended, real-valued, nonincreasing function defined from $[0, \infty)$ to $[0, \infty]$. As before, let $D = d(0)$ denote the least upper bound on the demand, but now D is possibly infinite $(0 < D \leqslant \infty)$. Let $D_{\min} = \lim_{x\to\infty} d(x)$ be the greatest lower bound on the demand with $0 \leqslant D_{\min} < D \leqslant \infty$. We assume that $d(\cdot)$ is finite, strictly decreasing, and twice differentiable over $(x_{\min}, x_{\max})$ where $x_{\min} = \sup\{x \geqslant 0: d(x) = D\}$ and $x_{\max} = \inf\{x \geqslant 0: d(x) = D_{\min}\}$. Hence, $d'(x) < 0$ and $d(x) < \infty$ for $x_{\min} < x < x_{\max}$, where $d'(\cdot)$ is the derivative of $d(\cdot)$. Under these conditions, we can restate A3 as follows:

ASSUMPTION A3. $e(x) = -xd'(x)/d(x)$ is (strictly) increasing for $x \in (x_{\min}, x_{\max})$, where $e(\cdot)$ is the price elasticity of the demand function.

Then, the conclusions of Proposition 5.1 immediately follow after deriving the equivalent conditions given in Lemma A.2.

Note that Proposition 2.1 also holds under this general setting once A4 is restated as

ASSUMPTION A4. $\bar{e}(y) = -yp'(y)/p(y)$ is (strictly) increasing for $y \in (D_{\min}, D)$, where $p(y)$ is assumed to be finite for $y \in (D_{\min}, D)$ and $p'(\cdot)$ is the derivative of $p(\cdot)$.

8. Conclusions

We showed that none of Assumptions A1, A2, and A3 can be claimed to be more restrictive than any other. However, when restricted to certain parameter regions, we showed that the assumptions can be ordered from the strongest to the weakest. This gives a preferred order for the assumptions if they are required to hold only over these restricted regions.

Our results also have some economic implications. Because $e(x) \leqslant 1$ for $x \in A_p$ and $e(p(y)) \leqslant 1$ for $y \in A_d$, A_p and A_d are, respectively, the set of prices and set of demand values for which the demand is inelastic (or unit elastic for x^*). Similarly, A_p^c and A_d^c are, respectively, the set of prices and set of demand values for which the demand is elastic. Then, from Proposition 5.1, we know that over the region where demand is inelastic, decreasing marginal revenue with respect to demand implies increasing price elasticity, which in turn implies decreasing marginal revenue with respect to price. On the other hand, over the

region where demand is elastic, decreasing marginal revenue with respect to price implies increasing price elasticity, which in turn implies decreasing marginal revenue with respect to demand. Finally, from the examples, we also know that none of these implications holds in the opposite direction.

Appendix

PROOF OF PROPOSITION 2.1. It can easily be shown that

$$e(x) = \frac{-xd'(x)}{d(x)},$$

where $d'(x)$ is the derivative of $d(x)$ with respect to x. Let $\bar{e}(\cdot)$ denote the generalized failure rate function for $H(\cdot)$. Then, similar to $e(\cdot)$, we have

$$\bar{e}(y) = \frac{-yp'(y)}{p(y)},$$

where $p'(y)$ is the derivative of $p(y)$ with respect to y. Using the fact that $d(p(y)) = y$, it follows that $p'(y) = 1/d'(p(y))$. Letting $x = p(y)$, we get

$$\bar{e}(y) = \frac{-d(x)}{xd'(x)} = \frac{1}{e(x)}.$$

Because x decreases as y increases (and vice versa), we conclude that A3 and A4 are equivalent conditions. □

LEMMA A.1. *Let $a \in R$ and $b \in R$ be such that $x_{\min} \leqslant a < b < x_{\max}$. Then,*

(i) $R_d(y)$ is (strictly) concave for $y \in [d(b), d(a)]$ if and only if $2h(x) \geqslant (>) -f'(x)/f(x)$ for $x \in (a, b)$.

(ii) $R_p(x)$ is (strictly) concave for $x \in [a, b]$ if and only if $2/x \geqslant (>) -f'(x)/f(x)$ for $x \in (a, b)$.

(iii) $e(x)$ is (strictly) increasing for $x \in [a, b]$ if and only if $1/x + h(x) \geqslant (>) -f'(x)/f(x)$ for $x \in (a, b)$.

PROOF. First, note that when $x \in (a, b)$, we have $x > 0$, $f(x) > 0$, and $F(x) < 1$ because $0 \leqslant x_{\min} \leqslant a < b < x_{\max}$.

Part (ii) immediately follows after taking the second derivative of $R_p(x)$ with respect to x and part (iii) follows after taking the first derivative of $e(x)$.

For part (i), first let $F^{-1}(\cdot) = G(\cdot)$. Then, $G(z)$ is differentiable for $0 < z < 1$ because $F(\cdot)$ is differentiable and $f(G(z)) > 0$ for $0 < z < 1$ (e.g., see Edwards and Penney 1990). Let $G'(\cdot)$ be the first derivative of $G(\cdot)$. Using $F(G(z)) = z$, we get $f(G(z))G'(z) = 1$ and it follows that

$$G'(z) = \frac{1}{f(G(z))} \quad \text{for } 0 < z < 1. \tag{6}$$

Then, using (6), the fact that $G(z)$ is differentiable for $0 < z < 1$ and the chain rule, we conclude that $G'(z)$ is differentiable for $0 < z < 1$. Let $G''(z)$ denote the second derivative of $G(\cdot)$. Then, using (6), we get

$$G''(z) = -\frac{f'(G(z))}{(f(G(z)))^3} \quad \text{for } 0 < z < 1. \tag{7}$$

Now, twice differentiating $R_d(y)$ with respect to y, using (6) and (7) for $0 < y < D$, we get the following:

$$\frac{d^2(R_d(y))}{dy^2} = 2\frac{dp(y)}{dy} + y\frac{d^2(p(y))}{dy^2}$$

$$= -\frac{2}{y}G'\left(1 - \frac{y}{D}\right) + \frac{y}{D^2}G''\left(1 - \frac{y}{D}\right)$$

$$= \frac{1}{D}\left(\frac{-2}{f(G(1-y/D))} - \frac{y}{D}\frac{f'(G(1-y/D))}{(f(G(1-y/D)))^3}\right).$$

Let $x = G(1 - y/D)$. Then, $y/D = 1 - F(x)$ and we have

$$\frac{d^2(R_d(y))}{dy^2} = \frac{1}{D}\left(-\frac{2}{f(x)} - \frac{(1-F(x))f'(x)}{(f(x))^3}\right) \tag{8}$$

$$= \frac{1}{D}\left(\frac{-2(f(x))^2 - (1-F(x))f'(x)}{(f(x))^3}\right). \tag{9}$$

Because $f(x) > 0$, $d^2(R_d(y))/dy^2 \leqslant (<)0$ if and only if $2(f(x))^2 + (1-F(x))f'(x) \geqslant (>)0$, which is equivalent to $2h(x) \geqslant (>) -f'(x)/f(x)$. (Note that conditions with the strict inequalities $(>)$ correspond to the "strict" version of the assumption.) □

PROOF OF PROPOSITION 5.1. In the following, we prove the "strict" versions of the implications. The proofs of "nonstrict" versions follow similarly.

First, note that under any of Assumptions A1, A2, and A3, it can be shown that $e(x) \leqslant 1$ $(h(x) \leqslant 1/x)$ for $x \in A_p$, while $e(x) \geqslant 1$ $(h(x) \geqslant 1/x)$ for $x \in A_p^c$. Note also that $x > 0$, $f(x) > 0$, and $F(x) < 1$, if $x \in A_p$ or $x \in A_p^c$.

(i) Let $y \in A_d$ and $x = p(y)$. Then, $h(x) \leqslant 1/x$ because $x \in A_p$. It follows that

$$\frac{1}{x} + h(x) \geqslant 2h(x) > -\frac{f'(x)}{f(x)},$$

where the second inequality follows from Lemma A.1(i). Then, the result follows from Lemma A.1(iii).

(ii) Let $x \in A_p$. Then, $h(x) \leqslant 1/x$. It follows that

$$\frac{2}{x} \geqslant \frac{1}{x} + h(x) > -\frac{f'(x)}{f(x)},$$

where the second inequality follows from Lemma A.1(iii). Then, the result follows from Lemma A.1(ii).

(iii) Let $x \in A_p^c$. Then, $h(x) \geqslant 1/x$. It follows that

$$\frac{1}{x} + h(x) \geqslant \frac{2}{x} > -\frac{f'(x)}{f(x)},$$

where the second inequality follows from Lemma A.1(ii). Then, the result follows from Lemma A.1(iii).

(iv) Let $x \in A_p^c$. Then, $h(x) \geqslant 1/x$. It follows that

$$2h(x) \geqslant \frac{1}{x} + h(x) > -\frac{f'(x)}{f(x)},$$

where the second inequality follows from Lemma A.1(iii). Then, the result follows from Lemma A.1(i). □

LEMMA A.2. *Let $a \in R$ and $b \in R$ be such that $x_{\min} \leqslant a < b < x_{\max}$. Then,*

(i) $R_d(y)$ is (strictly) concave for $y \in [d(b), d(a)]$ if and only if $d'(x)(e(x) - 1) - e'(x)d(x) \leqslant (<)0$ for $x \in (a, b)$.

(ii) $R_p(x)$ is (strictly) concave for $x \in [a, b]$ if and only if $d'(x)(1 - e(x)) - e'(x)d(x) \leqslant (<)0$ for $x \in (a, b)$.

(iii) $e(x)$ is (strictly) increasing for $x \in [a, b]$ if and only if $e'(x) \geqslant (>)0$ for $x \in (a, b)$, where $d'(\cdot)$ and $e'(\cdot)$ are the derivatives of $d(\cdot)$ and $e(\cdot)$, respectively.

PROOF. First, note that when $x \in (a, b)$, we have $x > 0$, $d'(x) < 0$, and $0 < d(x) < \infty$ because $0 \leqslant x_{\min} \leqslant a < b < x_{\max}$.

(i) We prove the "strict" version of the result. The "non-strict" version follows similarly.

Similar to the proof of Lemma A.1(i), using the fact that $d(p(y)) = y$, we can show that $p'(y) = 1/d'(p(y))$ and $p''(y) = -d''(p(y))/(d'(p(y)))^3$, where $p'(\cdot)$ and $p''(\cdot)$ are the first and second derivatives of $p(\cdot)$, and $d'(\cdot)$ and $d''(\cdot)$ are the first and second derivatives of $d(\cdot)$, respectively. Then, letting $R_d''(\cdot)$ denote the second derivative of $R_d(\cdot)$, we can show that

$$R_d''(y) = \frac{2}{d'(x)} - \frac{d(x)d''(x)}{(d'(x))^3}, \quad \text{where } x = p(y).$$

Because $d'(x) < 0$ for $x_{\min} < x < x_{\max}$, it follows that $R_d''(y) < 0$ if and only if $2(d'(x))^2 - d(x)d''(x) > 0$ for $x = p(y)$. Using the fact that $x_{\min} < x < x_{\max}$, we know that $x > 0$ and $0 < d(x) < \infty$. Then, for $x = p(y)$, we can proceed as follows:

$$R_d''(y) < 0$$

$$\Leftrightarrow 2(d'(x))^2 - d(x)d''(x) > 0$$

$$\Leftrightarrow xd(x)d''(x) - 2x(d'(x))^2 < 0$$

$$\Leftrightarrow \frac{d(x)d'(x) + xd(x)d''(x) - x(d'(x))^2 - x(d'(x))^2 - d(x)d'(x)}{(d(x))^2} < 0$$

$$\Leftrightarrow -e'(x) - \frac{x(d'(x))^2 + d(x)d'(x)}{(d(x))^2} < 0$$

$$\Leftrightarrow -e'(x) + e(x)\frac{d'(x)}{d(x)} - \frac{d'(x)}{d(x)} < 0$$

$$\Leftrightarrow d'(x)(e(x) - 1) - e'(x)d(x) < 0.$$

(ii) Let $R_p'(\cdot)$ and $R_p''(\cdot)$ denote the first and second derivatives of $R_p(\cdot)$, respectively. Then, we have $R_p'(x) = d(x) + xd'(x) = d(x)(1 - e(x))$ from which it follows that $R_p''(x) = d'(x)(1 - e(x)) - d(x)e'(x)$. Hence, the result follows. $\square$

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