Resource-based Patient Prioritization in Mass-Casualty Incidents

Online Supplement

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A. Proofs of Analytical Results

Proof of Proposition 1. It is sufficient to show that there exists an optimal solution where only one class is served at a time almost everywhere. That is, the set of points over which more than one point is served simultaneously will have measure zero. Because changing the solution over a set of points with measure zero does not change the value of the objective function in (P1), we can change the solution at these points so that only one class is served.

Denote the total expected reward associated with any solution S of (P1) by $z(S)$. Consider any solution S such that there exist two classes of patients (without loss of generality, classes 0 and 1) served simultaneously over some set of positive measure. Denote by $Y \subseteq [0,T]$ the set of all points where classes 0 and 1 are served simultaneously under S. We will construct a solution $\bar{S}$ such that $z(\bar{S}) \geq z(S)$ but in which the set of points where classes 0 and 1 are served simultaneously has measure zero.

Y can be partitioned into $Y_0 \cup Y_1 \cup \cdots$, where Y_0 is a set of points of measure zero, and $\forall j \in \{1,2,\ldots\}$, Y_j is an open interval such that where *either* $f_1'(t) \leq f_0'(t)$ or $f_1'(t) \geq f_0'(t)$ for all $t \in Y_j$, *and* $r_i(t)$ is continuous over $t \in Y_j$ for $i = 0,1$. Now, take any of the open intervals, $Y_j = (a,b)$, where $0 \leq a < b \leq T$ and $f_1'(t) \geq f_0'(t)$ for $t \in (a,b)$. That is, the reward gap function $g_{1,0}(t) = f_1(t) - f_0(t)$ is non-decreasing for $t \in (a,b)$. (The case where $f_1'(t) \leq f_0'(t)$ is symmetric, and hence its proof is omitted.)

Because $r_0(t)$ and $r_1(t)$ are continuous over (a,b), there must exist $c \in (a,b)$ such that

$$\int_a^c r_1(t)dt = \int_c^b r_0(t)dt. \tag{A-1}$$

In particular, such c must exist because $\lim_{c\to a}\int_a^c r_1(t)dt = 0$, $\lim_{c\to b}\int_c^b r_0(t)dt = 0$, the left-hand side of (A-1) is non-decreasing in c (because $r_1(t) \geq 0$), and the right-hand side of (A-1) is non-increasing in c (because $r_0(t) \geq 0$).

To construct $\bar{S}$, change the service during (a,b) as follows: during (a,c) serve class 0 at rate $r_0(t) + r_1(t)$, and during (c,b) serve class 1 at rate $r_0(t) + r_1(t)$. Note that the total amount of service to each class within (a,b) is unchanged, hence the constraints in (P1) are satisfied for $\bar{S}$. Now, we have

$$\begin{aligned}
z(\bar{S}) &= z(S) + \int_a^c r_1(t)f_0(t)dt + \int_c^b r_0(t)f_1(t)dt - \int_a^c r_1(t)f_1(t)dt - \int_c^b r_0(t)f_0(t)dt \\
&= z(S) + \int_c^b r_0(t)g_{1,0}(t)dt - \int_a^c r_1(t)g_{1,0}(t)dt \\
&\geq z(S) + g_{1,0}(c)\left(\int_c^b r_0(t)dt - \int_a^c r_1(t)dt\right) = z(S).
\end{aligned}$$

Here the inequality follows because $r_i(t) \geq 0$ for $t \in (a,b)$ and $i = 0,1$, and $g_{1,0}(t)$ is non-decreasing, which implies that $g_{1,0}(c) \geq g_{1,0}(t)$ for all $t \in [c,b]$ and $g_{1,0}(c) \leq g_{1,0}(t)$ for all $t \in [a,c]$. Finally, the last equation follows by (A-1). We can then repeat this construction for the remaining intervals over which more than one class is served under S. We conclude that for any solution that serves more than one class at any given time, there is another solution that performs at least as well by serving only one class at any point in time. The result immediately follows. □

Proof of Proposition 2. Consider a solution $\mathbf{W}$, where class i is served at least partially before class j. Let $Y \subseteq W(i)$ be the set of all points such that $\forall s \in Y, \exists t \in W(j)$ where $s < t$. That is, Y is the set of points where i is served before j. Similarly, let $Z \subseteq W(j)$ be the set of points such that $\forall s \in Z, \exists t \in W(i)$ where $s > t$. That is, Z is the set of all points where j is served after i.
Partition $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_{m_i}$, where $1 \leq m_i < \infty$, such that Y_0 is a set of points having measure zero; $Y_k, k = 1, 2, \ldots, m_i$, are open intervals; and $p > q \implies \forall s \in Y_p, t \in Y_q : s > t$. Similarly, partition $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_{m_j}$, where $1 \leq m_j < \infty$, such that Z_0 is a set of points having measure zero; $Z_k, k = 1, 2, \ldots, m_j$, are open intervals; and $p > q \implies \forall s \in Z_p, t \in Z_q : s > t$. The fact that m_i and m_j are finite follows from our assumption that $W(i)$ contains only finitely many intervals.
Let $Y_1 \equiv (a_i, b_i)$ and $Z_{m_j} \equiv (a_j, b_j)$, where $0 \leq a_i < b_i \leq a_j < b_j \leq T$. Let $\epsilon = \min\{b_i - a_i, b_j - a_j\}$. We will define a new solution $\bar{\mathbf{W}}$ that performs at least as well as $\mathbf{W}$:

$$\begin{aligned}\bar{W}(i) &= (W(i) \backslash (a_i, a_i + \epsilon)) \cup (b_j - \epsilon, b_j)\\ \bar{W}(j) &= (W(j) \backslash (b_j - \epsilon, b_j)) \cup (a_i, a_i + \epsilon)\\ \bar{W}(k) &= W(k), \quad \forall k \in \mathcal{I} \backslash \{i, j\}.\end{aligned}$$

Since the constraint set of (P2) is satisfied for $\mathbf{W}$, and the construction of $\bar{\mathbf{W}}$ does not change the measure of any of the solution sets, except by adding and subtracting sets of the same measure (ϵ), then the constraints of (P2) are satisfied for $\bar{\mathbf{W}}$.
Now, let $g(t) = f_j(t) - f_i(t)$, for all $t \in [0, T]$, and let $z(\mathbf{W})$ be the total expected reward obtained from using solution $\mathbf{W}$. Then,

$$\begin{aligned}z(\bar{\mathbf{W}}) &= z(\mathbf{W}) - \int_{a_i}^{a_i+\epsilon} f_i(t)\,dt + \int_{b_j-\epsilon}^{b_j} f_i(t)\,dt - \int_{b_j-\epsilon}^{b_j} f_j(t)\,dt + \int_{a_i}^{a_i+\epsilon} f_j(t)\,dt\\ &= z(\mathbf{W}) + \int_{a_i}^{a_i+\epsilon} (f_j(t) - f_i(t))\,dt - \int_{b_j-\epsilon}^{b_j} (f_j(t) - f_i(t))\,dt \geq z(\mathbf{W}) + \epsilon g(a_i + \epsilon) - \epsilon g(b_j - \epsilon)\\ &\geq z(\mathbf{W}),\end{aligned}$$

where the first inequality holds because $f_j'(t) \leq f_i'(t)$, and hence $g'(t) \leq 0$ for all $t \in [0, T]$, and the second inequality holds because $a_i + \epsilon \leq b_i \leq a_j \leq b_j - \epsilon$ and $g'(t) \leq 0$ for all $t \in [0, T]$.
Note that for $\bar{\mathbf{W}}$ we guarantee that either Y or Z will have at least one fewer open interval than $\mathbf{W}$. Hence, we will be able to repeat this procedure at most $m_i + m_j$ times until Y and Z are of measure zero. At that point we can set the service of points in Y and Z arbitrarily, because sets of points of measure zero do not affect the expected total reward. Then, the resulting solution will have a reward at least as large as $z(\mathbf{W})$ but without any service of class i before class j. $\square$

Proof of Proposition 3. To prove the result, we show that in any optimal solution to (P3), $W(D)$ is a single interval, plus possibly a set of zero-measure points. Now, suppose this is not true, that is, in the optimal solution there are at least two intervals contained in $W(D)$ with non-zero measure, such that the points between these two intervals are not in $W(D)$. In other words, there exist $0 \leq a_1 < b_1 < a_2 < b_2 \leq T$ such that $(a_1, b_1) \cup (a_2, b_2) \subseteq W(D)$, but $(b_1, a_2) \not\subseteq W(D)$. We will show that such a solution cannot be optimal. We must have one of the following three cases:
Case 1 ($t_m \leq b_1$): Let $\bar{W}(D) \equiv (W(D) \backslash (a_2, b_2)) \cup (b_1, b_1 + b_2 - a_2)$. Then, if we let $z(\mathbf{W})$ be the reward obtained by using solution $\mathbf{W}$, we have

$$\begin{aligned}z(\bar{\mathbf{W}}) &= z(\mathbf{W}) + \int_{b_1}^{b_1+b_2-a_2} g(t)\,dt - \int_{a_2}^{b_2} g(t)\,dt = z(\mathbf{W}) + \int_{b_1}^{b_1+b_2-a_2} (g(t) - g(t + a_2 - b_1))\,dt\\ &> z(\mathbf{W}),\end{aligned}$$

implying that $\mathbf{W}$ is not optimal. Here, the inequality follows from the facts that $g(t)$ is decreasing in t for all $t > t_m$, $t_m \leq b_1$, and $b_1 < a_2 < b_2$.

Case 2 ($b_1 < t_m \le a_2$): Let $\bar{W}(D) \equiv (W(D) \backslash (a_1, a_1 + t_m - b_1)) \cup (b_1, t_m)$. Then, we have

$$z(\bar{\mathbf{W}}) = z(\mathbf{W}) + \int_{b_1}^{t_m} g(t)\,dt - \int_{a_1}^{a_1+t_m-b_1} g(t)\,dt = z(\mathbf{W}) + \int_{a_1}^{a_1+t_m-b_1} (-g(t) + g(t + b_1 - a_1))\,dt$$
$$> z(\mathbf{W}),$$

implying that $\mathbf{W}$ is not optimal. Here, the inequality follows from the facts that $g(t)$ is increasing in t for all $t < t_m$ and $a_1 < b_1 < t_m$.

Case 3 ($a_2 < t_m$): Let $\bar{W}(D) \equiv (W(D) \backslash (a_1, b_1)) \cup (a_1 + a_2 - b_1, a_2)$. Then, we have

$$z(\bar{\mathbf{W}}) = z(\mathbf{W}) + \int_{a_1+a_2-b_1}^{a_2} g(t)\,dt - \int_{a_1}^{b_1} g(t)\,dt = z(\mathbf{W}) + \int_{a_1}^{b_1} (-g(t) + g(t + a_2 - b_1))\,dt > z(\mathbf{W}),$$

implying that $\mathbf{W}$ is not optimal. Here, the inequality follows from the facts that $g(t)$ is increasing in t for all $t < t_m$, $a_2 < t_m$, and $a_1 < b_1 < a_2$. □

Proof of Proposition 4. (i) By the Fundamental Theorem of Calculus and the fact that $g(t)$ is continuous,

$$v'(t) = g(t + n_D) - g(t), \text{ and} \tag{A-2}$$

$$v''(t) = g'(t + n_D) - g'(t), \text{ for } t \ge 0, \tag{A-3}$$

where $v''(\cdot)$ is the second derivative of $v(t)$. Now, by Assumption 1, we have $v'(t) < 0$ for $t \ge t_m$, and if $t_m \ge n_D$, then $v'(t) > 0$ for $t \le t_m - n_D$. Hence, any global maximizer of $v(t)$ over $[0, \infty)$ must be in $[\max\{0, t_m - n_D\}, t_m]$.

We next show that $\tilde{t}$ is unique. Note that for all $t \in (\max\{0, t_m - n_D\}, t_m)$, $g'(t) > 0$ and $g'(t+n_D) < 0$. Then from Equation (A-3), we have $v''(t) < 0$ for $t \in (\max\{0, t_m - n_D\}, t_m)$. Hence, if $t_m > 0$, then there is a unique maximizer of $v(t)$ in $(\max\{0, t_m - n_D\}, t_m)$. Otherwise, $t_m = 0$ is the unique maximizer of $v(t)$. In summary, there is a unique maximizer $\tilde{t} = \arg \max_{t \in [0,\infty)} v(t)$, and we have $v'(t) > 0$ for $t < \tilde{t}$ and $v'(t) < 0$ for $t > \tilde{t}$.

(ii) If $\tilde{t} \le n_I$, then $\tilde{t}$ is also the global maximizer of $v(t)$ for the domain $[0, n_I]$, i.e., $t^* = \tilde{t}$. Otherwise, because $v'(t) > 0$ for $t < \tilde{t}$, we have $v(n_I) > v(t)$ for all $t < n_I$. Hence, n_I is the global maximizer of $v(t)$ for the domain $[0, n_I]$, i.e., $t^* = n_I$.

(iii) Because $t^* \le \tilde{t}$ by part (ii) and $\tilde{t} \le t_m$ by part (i), we have $t^* \le t_m$. Moreover, Assumption 1 states that $t_m \le n_I + n_D$. Using this assumption and the fact that $t_m \le \tilde{t} + n_D$ by part (i), we conclude that $t_m \le \min\{n_I + n_D, \tilde{t} + n_D\} = n_D + \min\{n_I, \tilde{t}\} = n_D + t^*$, where the last equation is due to part (ii). □

Proof of Theorem 1. From Proposition 4, we know that $\tilde{t}$ exists, is unique, and is in the interval $[\max\{0, t_m - n_D\}, t_m]$. Because $\tilde{t}$ is a maximizer of $v(t)$ over $[0, \infty)$, and $v'(t) = g(t + n_D) - g(t)$ is defined for all $t \ge 0$, then either $\tilde{t}$ is a stationary point (i.e., it satisfies $v'(\tilde{t}) = 0$), or $v'(0) < 0$ and hence $\tilde{t} = 0$. The latter corresponds to case (i), where $t^* = 0$ by part (ii) of Proposition 4.

We now show that when $\tilde{t}$ is a stationary point of $v(t)$, then exactly one of statements (ii) or (iii) must hold. Because $\tilde{t}$ is a stationary point, by definition $v'(\tilde{t}) = 0$, or in other words, $g(\tilde{t}) = g(\tilde{t} + n_D)$. Furthermore, from the proof of part (i) of Proposition 4, we know that $g(T) - g(n_I) = v'(n_I) \ge 0$ if and only if $\tilde{t} \ge n_I$. Part (ii) of Proposition 4 completes the proof. □

Proof of Proposition 5. Let $t^*(n_D)$ and $\tilde{t}(n_D)$ denote the values of t^* and $\tilde{t}$, respectively, when the number of class 1 patients is n_D. We first show that $\tilde{t}(n_D)$ either stays at zero (case 1) or decreases with n_D (case 2).

Case 1 ($\tilde{t}(n_D) = 0$): We will show that for any $\bar{n}_D > n_D$, $\tilde{t}(\bar{n}_D) = 0$. From Theorem 1, we know that $g(0) \ge g(n_D)$, and hence by Assumption 1 and the assumption that $n_D > 0$, we have $t_m < n_D$, which implies that $g(n_D) > g(\bar{n}_D)$. Hence, $g(0) > g(\bar{n}_D)$, which by Theorem 1 yields that $\tilde{t}(\bar{n}_D) = 0$.

Case 2 ($\tilde{t}(n_D) > 0$): We will show that for any $\bar{n}_D > n_D$, $\tilde{t}(\bar{n}_D) < \tilde{t}(n_D)$. From Proposition 4, we know that $\tilde{t}(\bar{n}_D) \le t_m$. Therefore, it is sufficient to show that $\tilde{t}(\bar{n}_D)$ cannot be in the interval $[\tilde{t}(n_D), t_m]$. To do this, take any $t \in [\tilde{t}(n_D), t_m]$. We will show that t cannot be the maximizer of $v(t)$ when there are $\bar{n}_D$ class 1 patients. We have $g(t) \ge g(\tilde{t}(n_D)) \ge g(\tilde{t}(n_D) + n_D) > g(\tilde{t}(n_D) + \bar{n}_D) \ge g(t + \bar{n}_D)$, where the first inequality follows from the facts that $g'(s) > 0$ for $s < t_m$ and $t \le t_m$; the second inequality follows from Theorem 1; the third inequality follows from the facts that $g'(s) < 0$ for $s > t_m$, $\tilde{t}(n_D) + n_D \ge t_m$ by Proposition 4, and $\bar{n}_D > n_D$; and the final inequality follows from the facts that $g'(s) < 0$ for $s > t_m$ and $t \ge \tilde{t}(n_D)$. Hence, we conclude that $g(t) > g(t + \bar{n}_D)$ for any $t \in [\tilde{t}(n_D), t_m]$. However, Theorem 1 implies that the unique maximizer of $v(s)$ over $s \in [0, \infty)$ is either equal to zero or is a stationary point, which satisfies $g(s) = g(s + \bar{n}_D)$ when there are $\bar{n}_D$ class 1 patients. Since in this case, neither one of these holds for any $t \in [\tilde{t}(n_D), t_m]$, we conclude that $t \in [\tilde{t}(n_D), t_m]$ cannot be the maximizer of $v(s)$ over $s \in [0, \infty)$ when there are $\bar{n}_D$ class 1 patients. We showed that $\tilde{t}(n_D)$ either stays at zero or decreases with n_D. Thus, $t^*(n_D)$ either decreases or stays the same (at n_I or zero) as n_D increases since $t^*(n_D) = \min\{\tilde{t}(n_D), n_I\}$ by part (ii) of Proposition 4. □

B. Estimating Reward Functions

We estimated the survival probability function for a given class $i \in \{I, D\}$ by

$$f_i(t) = \sum_{j=0}^{12} \pi_i(j) s_j(t) \text{ for } i \in \{I, D\}, \tag{A-4}$$

where $s_j(t)$ is the probability that a patient with RPM value $j \in \{0, 1, \ldots, 12\}$ ultimately survives if he or she is transported at time t and $\pi_i(j)$, for $i \in \{I, D\}$ and $j \in \{0, 1, \ldots, 12\}$, is the probability that a randomly selected patient who is in START class i would have an RPM score of j. (RPM can take any integer value between 0 and 12, with lower values indicating more critical conditions.) The survival probability functions $s_j(\cdot)$, for $j \in \{0, 1, \ldots, 12\}$, were estimated in Sacco et al. (2005, 2007) for three different types of injuries, and in this article, we use the estimates for penetrating injuries provided in Sacco et al. (2007). To obtain estimates for $\pi_i(j)$, for $j \in \{0, 1, \ldots, 12\}$, we consulted Prof. Winslow (Winslow et al. 2010), who informed us that his estimates for the distribution of the START class of a patient given his/her RPM score would be more reliable than those for the RPM score distribution given a patient's START class (i.e., $\pi_i(j)$, for $j \in \{0, 1, \ldots, 12\}$). Hence, we expressed $\pi_i(j)$, using Bayes' Law, as follows:

$$\pi_i(j) = \frac{q_j p_j(i)}{\sum_{k=0}^{12} q_k p_k(i)}, \text{ for } i \in \{I, D\} \text{ and } j \in \{0, 1, \ldots, 12\}, \tag{A-5}$$

where q_j is the probability that a randomly chosen patient has an RPM score of $j \in \{0, 1, \ldots, 12\}$ and $p_j(i)$ is the probability that a patient with an RPM score of $j \in \{0, 1, \ldots, 12\}$ belongs to START-class of $i \in \{E, I, D, M\}$. The probabilities $p_j(i)$ were estimated by Prof. Winslow. The only remaining estimate we need is for q_j, for $j \in \{0, 1, \ldots, 12\}$. It is likely that this distribution varies depending on the type of injuries and event. Therefore, in our simulation study, we systematically considered different probability distributions for the initial RPM score of a patient (q_j). For each distribution, we determined the corresponding survival probability functions for both immediate and delayed patients using (A-4).

More specifically, we assumed that the initial RPM scores of the patients came from a discretized and rescaled version of the Beta distribution, which is typically used in the absence of data. We considered five scenarios using five different Beta distributions with the two parameters (α_1, α_2) given by (1.5,5), (1.5,3), (1,1), (3,1.5), and (5,1.5). As we go from the first scenario to the last, the distribution changes from being right-skewed to left-skewed. Scenario 3, where $\alpha_1 = \alpha_2 = 1$, is the case where RPM scores are uniformly distributed. The corresponding empirical survival probability functions for immediate and delayed patients as well as the probability distributions for the START classes are provided in Figure 3.

For each of the five scenarios we constructed above, we fit the immediate and delayed reward functions ($f_I(t)$ and $f_D(t)$, respectively) to the three-parameter function given by (3) using Matlab's `nlinfit` function, which performs a nonlinear least-squares regression. The fitted parameters for each scenario are given in Table A-1.

Table A-1 Fitted parameters for the five survival scenarios.

	Immediate			Delayed		
Scenario	$\beta_{0,I}$	$\beta_{1,I}$	$\beta_{2,I}$	$\beta_{0,D}$	$\beta_{1,D}$	$\beta_{2,D}$
1	0.09	17	1.01	0.57	61	2.03
2	0.15	28	1.38	0.65	86	2.11
3	0.24	47	1.30	0.76	138	2.17
4	0.40	59	1.47	0.77	140	2.29
5	0.56	91	1.58	0.81	160	2.41

Figure 3 demonstrates visually that the log-logistic model given by (3) fits well to the empirical data. In order to quantify the goodness of fit to the data, we also estimated the mean-squared error of this model as well as three other commonly used models. These estimates for immediate and delayed patients under the five scenarios considered in our simulation study are presented in Table A-2. From the mean-squared error estimates, we observe that except for the exponential model, all models (namely, the log-logistic, log-normal, and Weibull models) provided a reasonable fit having mean-squared errors below 6×10^{-4} for each patient type and under each scenario. Although there does not appear to be a clear contender among these three models, the log-logistic model provided the smallest value of the worst-case mean-squared error across all patient types and scenarios.

Table A-2 Mean-squared errors for four different models for the empirical survival probability data provided in Figure 3.

Immediate Survival Probability Functions ($\times 10^{-4}$)				
Scenario	Log-Logistic	Log-Normal	Weibull	Exponential
1	0.040	0.054	0.076	0.343
2	0.049	0.070	0.133	0.396
3	0.077	0.047	0.122	1.059
4	0.487	0.281	0.227	0.923
5	1.678	1.423	0.454	0.441

Delayed Survival Probability Functions ($\times 10^{-4}$)				
Scenario	Log-Logistic	Log-Normal	Weibull	Exponential
1	2.078	1.340	0.424	1.999
2	3.224	2.570	0.674	5.185
3	3.973	4.470	2.852	14.437
4	4.353	4.916	3.250	19.007
5	4.722	5.635	4.900	29.255