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Online Supplement

Appendix A - Proofs for Sections 3 and 4.1

Proof of Proposition 1: Let μ_{R1} and μ_{R2} denote the mean reservation prices for customers who are in the target and non-target segments of the regular product, respectively. Similarly, define μ_{P1} and μ_{P2} to be the mean reservation prices for customers who are in the target and non-target segments of the promotional product, respectively. Then, we have

$$E(X) = q_{R1}\mu_{R1} + q_{R2}\mu_{R2}$$

and

$$E(Y) = q_{P1}\mu_{P1} + q_{P2}\mu_{P2}.$$

We also have

$$\begin{aligned}
E(XY) &= q_{R1}\int_0^\infty\int_0^\infty xyf_{R1}(x)(\delta_{11}f_{P1}(y)+\delta_{12}f_{P2}(y))dxdy \\
&+ q_{R2}\int_0^\infty\int_0^\infty xyf_{R2}(x)(\delta_{21}f_{P1}(y)+\delta_{22}f_{P2}(y))dxdy \\
&= q_{R1}\int_0^\infty y(\delta_{11}f_{P1}(y)+\delta_{12}f_{P2}(y))\int_0^\infty xf_{R1}(x)dxdy \\
&+ q_{R2}\int_0^\infty y(\delta_{21}f_{P1}(y)+\delta_{22}f_{P2}(y))\int_0^\infty xf_{R2}(x)dxdy \\
&= q_{R1}\mu_{R1}\int_0^\infty y(\delta_{11}f_{P1}(y)+\delta_{12}f_{P2}(y))dy + q_{R2}\mu_{R2}\int_0^\infty y(\delta_{21}f_{P1}(y)+\delta_{22}f_{P2}(y))dy
\end{aligned}$$

Hence, $E(XY)$ can simply be written as

$$E(XY) = q_{R1}\mu_{R1}(\delta_{11}\mu_{P1}+\delta_{12}\mu_{P2}) + q_{R2}\mu_{R2}(\delta_{21}\mu_{P1}+\delta_{22}\mu_{P2}).$$

Then, after a few algebraic manipulations, it can be shown that

$$Cov(X,Y) = q_{R1}\mu_{R1}[\mu_{P1}(\delta_{11}-q_{P1})+\mu_{P2}(\delta_{12}-q_{P2})]+q_{R2}\mu_{R2}[\mu_{P1}(\delta_{21}-q_{P1})+\mu_{P2}(\delta_{22}-q_{P2})].$$

Since $\delta_{11}+\delta_{12} = q_{P1}+q_{P2} = \delta_{21}+\delta_{22} = 1$, we have $\delta_{11}-q_{P1} = -(\delta_{12}-q_{P2})$ and $\delta_{21}-q_{P1} = -(\delta_{22}-q_{P2})$, which yields

$$\begin{aligned}
Cov(X,Y) &= q_{R1}\mu_{R1}(\mu_{P1}-\mu_{P2})(\delta_{11}-q_{P1}) + q_{R2}\mu_{R2}(\mu_{P1}-\mu_{P2})(\delta_{21}-q_{P1}) \\
&= (\mu_{P1}-\mu_{P2})[q_{R1}\mu_{R1}(\delta_{11}-q_{P1}) + q_{R2}\mu_{R2}(\delta_{21}-q_{P1})]
\end{aligned}$$

Finally, using the fact that $q_{P1} = q_{R1}\delta_{11} + q_{R2}\delta_{21}$ (see (1)), we establish that

$$Cov(X,Y) = q_{R1}q_{R2}(\mu_{P1} - \mu_{P2})(\mu_{R1} - \mu_{R2})(\delta_{11} + \delta_{22} - 1).$$

Since $\mu_{P1} - \mu_{P2} > 0$ and $\mu_{R1} - \mu_{R2} > 0$ (due to assumption $(A3)$), the result follows. $\square$

Proof of Proposition 2: From (3), we have

$$q_{P1} - \widehat{q}_{P1} = q_{P1} - \frac{q_{R1}\delta_{11}\overline{F}_{R1}(r) + q_{R2}\delta_{21}\overline{F}_{R2}(r)}{q_{R1}\overline{F}_{R1}(r) + q_{R2}\overline{F}_{R2}(r)}.$$

Then, it can be shown that $q_{P1} - \widehat{q}_{P1} > 0$ if and only if

$$q_{R1}(q_{P1} - \delta_{11})\overline{F}_{R1}(r) - q_{R2}(\delta_{21} - q_{P1})\overline{F}_{R2}(r) > 0.$$

From assumption $(A3)$, we have $\overline{F}_{R1}(r) > \overline{F}_{R2}(r)$ (since failure rate ordering implies usual stochastic ordering), and it can be checked from (1) that $q_{R1}(q_{P1} - \delta_{11}) = q_{R2}(\delta_{21} - q_{P1})$. Then, for the above inequality to hold true, we must have $q_{P1} > \delta_{11}$, which can be shown to be equivalent to $\delta_{11} + \delta_{22} < 1$ using (1). Hence, the result follows. $\square$

Proof of Theorem 1: Throughout the proof, recall that an optimal p_t and d_t will solve the optimization problem in (8).

Proof of (a): Lemma 4 proves that $z_1^*(y,t) > z_2^*(y,t)$. Let $\bar{p} = z_1^*(y,t)$ and $\bar{d} = z_1^*(y,t) - z_2^*(y,t)$. Then $(\bar{p}, \bar{d})$ is a feasible price-discount pair for the optimization problem in (8), since $z_1^*(y,t) > z_2^*(y,t)$. Furthermore, $\Pi_1(\bar{p}, \Delta(y,t)) \geq \Pi_1(p, \Delta(y,t))$, $\forall p$, and $\Pi_2(\bar{p} - \bar{d}, \Delta(y,t)) \geq \Pi_2(p, \Delta(y,t))$, $\forall p$, since $z_1^*(y,t)$ and $z_2^*(y,t)$ maximize $\Pi_1(\cdot, \Delta(y,t))$ and $\Pi_2(\cdot, \Delta(y,t))$, respectively. Hence the result follows.

Proof of (b): If $\delta_{11} + \delta_{22} = 1$, it is easy to show that $q_{P1} = \widehat{q}_{P1}$. This implies that $z_1^*(y,t) = z_2^*(y,t)$, and thus it is optimal to set $p_t = z_1^*(y,t)$ and $d_t = 0$.

Now, in the remaining of the proof, suppose that $\delta_{11} + \delta_{22} > 1$. Then, Lemma 4 proves that $z_1^*(y,t) < z_2^*(y,t)$. Now, first, suppose for a contradiction that there exists an optimal price $\overline{p}_t$ and optimal discount $\overline{d}_t$ where $\overline{d}_t > 0$. Then, we must have $\Pi_1(\overline{p}_t - \overline{d}_t, \Delta(y,t)) \leq \Pi_1(\overline{p}_t, \Delta(y,t))$. (Otherwise, we obtain a contradiction to the optimality of $\overline{p}_t$.) Hence, using Lemma 3(a), we have $\Pi_2(\overline{p}_t - \overline{d}_t, \Delta(y,t)) < \Pi_2(\overline{p}_t, \Delta(y,t))$, which yields a contradiction to the optimality of $\overline{d}_t > 0$. Hence it must be that $\overline{d}_t = 0$.

Now that we know $\overline{d}_t = 0$, observe from (8) and the definition of $z^*(y,t)$ (given by (11)) that $z^*(y,t)$ will maximize the firm's revenue. It remains to show that $z_1^*(y,t) \leq z^*(y,t) \leq z_2^*(y,t)$.

First, suppose for a contradiction that $z^*(y,t) < z_1^*(y,t)$. By definition of $z_1^*(y,t)$, we have $\Pi_1(z^*(y,t), \Delta(y,t)) < \Pi_1(z_1^*(y,t), \Delta(y,t))$. Now, using Lemma 3(a), we observe $\Pi_2(z^*(y,t), \Delta(y,t)) < \Pi_2(z_1^*(y,t), \Delta(y,t))$. The last two inequalities together yield a contradiction to the optimality of $z^*(y,t)$ for $\lambda_P \Pi_1(\cdot, \Delta(y,t)) + \lambda_R \beta_R \Pi_2(\cdot, \Delta(y,t))$. Hence, $z^*(y,t) \geq z_1^*(y,t)$.

Similarly, suppose for a contradiction that $z^*(y,t) > z_2^*(y,t)$. By definition of $z_2^*(y,t)$, we have $\Pi_2(z_2^*(y,t), \Delta(y,t)) \geq \Pi_2(z^*(y,t), \Delta(y,t))$, and by Lemma 3(b), we obtain $\Pi_1(z_2^*(y,t), \Delta(y,t)) > \Pi_1(z^*(y,t), \Delta(y,t))$. The last two inequalities together yield a contradiction to the optimality of $z^*(y,t)$, so we must have $z^*(y,t) \leq z_2^*(y,t)$. The result follows.

□

Proof of Proposition 3: The proposition follows from the more general result proven in Lemma 5. □

Lemma 1 *Define* $\theta_i(p,\Delta) = (p - \Delta)\overline{F}_{Pi}(p)$, $i = 1, 2$. *Let* $\eta_i(y,t) = \arg\max_p\{\theta_i(p, \Delta(y,t)\}$, $i = 1, 2$. *Let* $z_1^*(y,t)$, $z_2^*(y,t)$ *and* $z^*(y,t)$ *be as defined by (9), (10) and (11). Then:*
(a) $\eta_2(y,t) \leq \eta_1(y,t)$,
(b) $\frac{d\theta_i(p,\Delta)}{dp} > (=)(<)0$ *for* $p < (=)(>)\eta_i(y,t)$,
(c) $z_1^*(y,t) \in [\eta_2(y,t), \eta_1(y,t)]$, $z_2^*(y,t) \in [\eta_2(y,t), \eta_1(y,t)]$, $z^*(y,t) \in [\eta_2(y,t), \eta_1(y,t)]$.

Proofs of (a) and (b): It is not difficult to show that $\theta_i(p,\Delta)$ is strictly unimodal in p due to assumption $(A2)$. (See, for example, Lariviere and Porteus, 2001.) Hence, $\eta_i(y,t)$ must satisfy the first-order condition (FOC) for $\theta_i(p, \Delta(y,t))$ with respect to p. The FOC for $\theta_i(p, \Delta(y,t))$, $i = 1, 2$, with respect to p is given by:

$$\frac{d\theta_i(p,\Delta)}{dp} = \overline{F}_{Pi}(p) - (p-\Delta)f_{Pi}(p) = \overline{F}_{Pi}(p)\left(1 - (p-\Delta)\frac{f_{Pi}(p)}{\overline{F}_{Pi}(p)}\right) = 0, i = 1, 2, \quad \text{(A-2)}$$

Now the FOCs in (A-2) along with assumption $(A3)$ yields part (a). Part (b) follows directly from the unimodality of $\theta_i(p,\Delta)$ in p.

Proof of (c): Recall that $z_1^*(y,t) = \inf\{p^* : \Pi_1(p^*, \Delta(y,t)) \geq \Pi_1(p, \Delta(y,t)), \forall p\}$. From (2) and (6), note that $\Pi_1(p, \Delta(y,t)) = q_{P1}\theta_1(p, \Delta(y,t)) + q_{P2}\theta_2(p, \Delta(y,t))$. Since $\theta_i(p, \Delta(y,t))$ is unimodal in p and $\eta_2(y,t) \leq \eta_1(y,t)$, it follows that $\Pi_1(p, \Delta(y,t))$ is increasing in p in the region where $p < \eta_2(y,t)$ and decreasing in p in the region where $p > \eta_1(y,t)$. Hence, it must be that $z_1^*(y,t) \in [\eta_2(y,t), \eta_1(y,t)]$. The proofs of $z_2^*(y,t) \in [\eta_2(y,t), \eta_1(y,t)]$ and $z^*(y,t) \in [\eta_2(y,t), \eta_1(y,t)]$ are similar. □

Proof of Proposition 4 : We will consider two separate cases:

Case 1: $\delta_{11} + \delta_{22} < 1$ — From Theorem 1(a), we have that $p_t^*(y) = z_1^*(y,t)$. We know that $z_1^*(y,t)$ is increasing in $\Delta(y,t)$ (by Lemma 6). Now noting that $\Delta(y,t)$ is decreasing in y and increasing in t (by Proposition 3), we conclude that $p_t^*(y)$ is decreasing in y and increasing in t. As for $p_t^*(y) - d_t^*(y)$, we note from Theorem 1(a) that $p_t^*(y) - d_t^*(y) = z_2^*(y,t)$, and we know that $z_2^*(y,t)$ is increasing in $\Delta(y,t)$ (by Lemma 6). Since $\Delta(y,t)$ is decreasing in y and increasing in t (by Proposition 3), it follows that $p_t^*(y) - d_t^*(y)$ is decreasing in y and increasing in t.

Case 2: $\delta_{11} + \delta_{22} \geq 1$ — From Theorem 1(b), we have that

$$p_t^*(y) = p_t^*(y) - d_t^*(y) = z^*(y,t) := \inf\{p^* : \Pi(p^*, \Delta(y,t)) \geq \Pi(p, \Delta(y,t)), \forall p\}$$

where $\Pi(p, \Delta(y,t)) = \lambda_P \Pi_1(p, \Delta(y,t)) + \lambda_R \beta_R \Pi_2(p, \Delta(y,t))$. After noting that $z^*(y,t)$ is increasing in $\Delta(y,t)$ (by Lemma 6), the desired result follows since $\Delta(y,t)$ is decreasing in y and increasing in t (by Proposition 3). □

Lemma 2 *Suppose that* $\delta_{11} + \delta_{22} < (>)(=)1$. *Then,* $\frac{d(\Pi_1(p,\Delta(y,t)) - \Pi_2(p,\Delta(y,t)))}{dp} > (<)(=)0$ *for* $p \in (\eta_2(y,t), \eta_1(y,t))$.

Proof of Lemma 2: We have

$$\Pi_1(p, \Delta(y,t)) - \Pi_2(p, \Delta(y,t)) = (\beta_P(p) - \alpha(p))(p - \Delta(y,t)).$$

Then,

$$\begin{aligned}
\frac{d(\Pi_1(p,\Delta(y,t)) - \Pi_2(p,\Delta(y,t)))}{dp} &= (\beta_P(p) - \alpha(p)) + (\beta_P'(p) - \alpha'(p))(p - \Delta(y,t)) \\
&= (q_{P1} - \widehat{q}_{P1})(\overline{F}_{P1}(p) - (p - \Delta(y,t)) f_{P1}(p)) \\
&+ (q_{P2} - \widehat{q}_{P2})(\overline{F}_{P2}(p) - (p - \Delta(y,t)) f_{P2}(p)) \\
&= (q_{P1} - \widehat{q}_{P1}) \\
&\times \left[\overline{F}_{P1}(p) - (p - \Delta(y,t)) f_{P1}(p) - \overline{F}_{P2}(p) + (p - \Delta(y,t)) f_{P2}(p)\right]
\end{aligned}$$

From Proposition 2, we know that $q_{P1} - \widehat{q}_{P1} > (<)(=)0$ if $\delta_{11} + \delta_{22} < (>)(=)1$. Furthermore, observe that the term in brackets is $\frac{d\theta_1(p,\Delta)}{dp} - \frac{d\theta_2(p,\Delta)}{dp}$, which is strictly positive for $p \in (\eta_2(y,t), \eta_1(y,t))$ by Lemma 1(b). The result now follows. □

Lemma 3 *Let x_1 and x_2 be such that $\eta_2(y,t) < x_1 < x_2 < \eta_1(y,t)$. Suppose that $\delta_{11} + \delta_{22} <(>)(=)1$.*

(a) If $\Pi_1(x_1,\Delta) \geq (\leq)(=)\Pi_1(x_2,\Delta)$, then $\Pi_2(x_1,\Delta) > (<)(=)\Pi_2(x_2,\Delta)$.

(b) If $\Pi_2(x_1,\Delta) \leq (\geq)(=)\Pi_2(x_2,\Delta)$, then $\Pi_1(x_1,\Delta) < (>)(=)\Pi_1(x_2,\Delta)$

Proof of Lemma 3: Suppose $\delta_{11} + \delta_{22} < 1$. Then, by Lemma 2, we have $\Pi_1(x_1,\Delta) - \Pi_2(x_1,\Delta) < \Pi_1(x_2,\Delta) - \Pi_2(x_2,\Delta)$. In addition, suppose $\Pi_1(x_1,\Delta) \geq \Pi_1(x_2,\Delta)$. The last two inequalities together imply that $\Pi_2(x_1,\Delta) > \Pi_2(x_2,\Delta)$, which completes the proof of part (a) for $\delta_{11} + \delta_{22} < 1$. Symmetric arguments yield part (b). The proofs for $\delta_{11} + \delta_{22} > 1$ and $\delta_{11} + \delta_{22} = 1$ are similar. □

Lemma 4 *Suppose that $\delta_{11} + \delta_{22} < (>)(=)1$. Then, $z_1^*(y,t) > (<)(=)z_2^*(y,t)$.*

Proof of Lemma 4: We will prove the result for $\delta_{11} + \delta_{22} < 1$; the proofs of the other cases are similar. The proof is by contradiction. Suppose $z_1^*(y,t) < z_2^*(y,t)$. By definition of $z_2^*(y,t)$, we have

$$\Pi_2(z_1^*(y,t),\Delta(y,t)) < \Pi_2(z_2^*(y,t),\Delta(y,t)).$$

Then, applying Lemma 3(b) with $x_1 = z_1^*(y,t)$ and $x_2 = z_2^*(y,t)$, we have

$$\Pi_1(z_1^*(y,t),\Delta(y,t)) < \Pi_1(z_2^*(y,t),\Delta(y,t)),$$

which is a contradiction to the optimality of $z_1^*(y,t)$ for $\Pi_1(\cdot,\Delta(y,t))$. Hence, when $\delta_{11}+\delta_{22} < 1$, we must have $z_1^*(y,t) \geq z_2^*(y,t)$. It still remains to show that $z_1^*(y,t) \neq z_2^*(y,t)$. To that end, suppose for a contradiction that $z_1^*(y,t) = z_2^*(y,t)$. Note that we must have $\left.\frac{d\Pi_2(p,\Delta(y,t)}{dp}\right|_{p=z_2^*(y,t)} = 0$ (since $z_2^*(y,t)$ is an interior optimizer of $\Pi_2(\cdot,\Delta(y,t))$). Hence, by Lemma 2, we must have $\left.\frac{d\Pi_1(p,\Delta(y,t)}{dp}\right|_{p=z_1^*(y,t)} > 0$, which is a contradiction to the optimality of $z_1^*(y,t)$ for $\Pi_1(\cdot,\Delta(y,t))$. Therefore, we cannot have $z_1^*(y,t) = z_2^*(y,t)$, which concludes the proof for $\delta_{11} + \delta_{22} < 1$. □

Lemma 5 *Consider a slightly more general version of the DP formulation in (5):*

$$\begin{aligned} V_t(y) &= V_{t-1}(y) + \max_{p_t \in A, d_t \in B(p_t)} \{\lambda_P \beta_P(p_t)\,(p_t + V_{t-1}(y-1) - V_{t-1}(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p_t - d_t)\,(p_t - d_t + V_{t-1}(y-1) - V_{t-1}(y))\right]\} \end{aligned} \qquad \text{(A-3)}$$

where both A and B are non-negative sets of real numbers, B possibly depends on p_t and for any $d_t \in B(p_t)$, $p_t - d_t \geq 0$. Then, we have:
(a) $V_{t+1}(y+1) - V_{t+1}(y) \geq V_t(y+1) - V_t(y)$, *or* $\Delta(y+1,t) \geq \Delta(y+1,t-1)$.
(b) $V_{t+1}(y) - V_t(y) \geq V_{t+2}(y) - V_{t+1}(y)$
(c) $V_t(y+1) - V_t(y) \geq V_t(y+2) - V_t(y+1)$, *or* $\Delta(y+1,t-1) \geq \Delta(y+2,t-1)$.

Proof of Lemma 5: Following Bitran and Mondschein (1993), we will use an inductive argument on $k = y + t$. For $k = 0$, all inequalities hold. Now, assume that all inequalities hold for $y + t < k$. We will prove that they also hold for $y + t = k$.

Proof of (a): Since there exists an optimal solution, for some $p^o \in A$ and $d^o \in B(p^o)$, we have

$$\begin{aligned} V_{t+1}(y) &= V_t(y) + \lambda_P \beta_P(p^o)(p^o + V_t(y-1) - V_t(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_t(y-1) - V_t(y))\right] \end{aligned}$$

which can also be written as

$$\begin{aligned} V_{t+1}(y) - V_t(y) &= \lambda_P \beta_P(p^o)(p^o + V_t(y-1) - V_t(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_t(y-1) - V_t(y))\right]. \end{aligned} \tag{A-4}$$

Then, we also have

$$\begin{aligned} V_{t+1}(y+1) &\geq V_t(y+1) + \lambda_P \beta_P(p^o)(p^o + V_t(y) - V_t(y+1)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_t(y) - V_t(y+1))\right], \end{aligned}$$

which can also be written as

$$\begin{aligned} V_{t+1}(y+1) - V_t(y+1) &\geq \lambda_P \beta_P(p^o)(p^o + V_t(y) - V_t(y+1)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_t(y) - V_t(y+1))\right]. \end{aligned} \tag{A-5}$$

By the induction assumption for (c),

$$V_t(y) - V_t(y+1) \geq V_t(y-1) - V_t(y).$$

Using this in (A-5), we get

$$\begin{aligned} V_{t+1}(y+1) - V_t(y+1) &\geq \lambda_P \beta_P(p^o)(p^o + V_t(y-1) - V_t(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_t(y-1) - V_t(y))\right]. \end{aligned} \tag{A-6}$$

Finally, from (A-4) and (A-6), we conclude that

$$V_{t+1}(y+1) - V_{t+1}(y) \geq V_t(y+1) - V_t(y).$$

Proof of (b): Since there exists an optimal solution, for some $p^o \in A$ and $d^o \in B(p^o)$, we have

$$\begin{aligned} V_{t+2}(y) &= V_{t+1}(y) + \lambda_P \beta_P(p^o)(p^o + V_{t+1}(y-1) - V_{t+1}(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_{t+1}(y-1) - V_{t+1}(y))\right], \end{aligned}$$

which can also be written as

$$\begin{aligned} V_{t+2}(y) - V_{t+1}(y) &= \lambda_P \beta_P(p^o)(p^o + V_{t+1}(y-1) - V_{t+1}(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_{t+1}(y-1) - V_{t+1}(y))\right]. \end{aligned} \tag{A-7}$$

Then, we also have

$$\begin{aligned} V_{t+1}(y) &\geq V_t(y) + \lambda_P \beta_P(p^o)(p^o + V_t(y-1) - V_t(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_t(y-1) - V_t(y))\right], \end{aligned}$$

which can also be written as

$$\begin{aligned} V_{t+1}(y) - V_t(y) &\geq \lambda_P \beta_P(p^o)(p^o + V_t(y-1) - V_t(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_t(y-1) - V_t(y))\right]. \end{aligned} \tag{A-8}$$

By the induction assumption on (a),

$$V_t(y-1) - V_t(y) \geq V_{t+1}(y-1) - V_{t+1}(y).$$

Using this in (A-8), we get

$$\begin{aligned} V_{t+1}(y) - V_t(y) &\geq \lambda_P \beta_P(p^o)(p^o + V_{t+1}(y-1) - V_{t+1}(y)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_{t+1}(y-1) - V_{t+1}(y))\right]. \end{aligned} \tag{A-9}$$

Finally, from (A-7) and (A-9), we conclude that

$$V_{t+1}(y) - V_t(y) \geq V_{t+2}(y) - V_{t+1}(y).$$

Proof of (c): Following as in parts (a) and (b), we can show that for some $p^o \in A$ and $d^o \in B(p^o)$, we have

$$\begin{aligned} V_t(y+2) - V_{t-1}(y+1) &= \lambda_P \beta_P(p^o)(p^o + V_{t-1}(y+1) - V_{t-1}(y+2)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_{t-1}(y+1) - V_{t-1}(y+2))\right] \\ &+ V_{t-1}(y+2) - V_{t-1}(y+1) \\ &= (V_{t-1}(y+2) - V_{t-1}(y+1))(1 - \lambda_P \beta_P(p^o) - \lambda_R \beta_R \alpha(p^o - d^o)) \\ &+ \lambda_P \beta_P(p^o) p^o + \left[\lambda_R \beta_R \alpha(p^o - d^o)(p^o - d^o)\right]. \end{aligned} \tag{A-10}$$

and

$$\begin{aligned} V_{t+1}(y+1) - V_t(y) &\geq \lambda_P \beta_P(p^o)(p^o + V_t(y) - V_t(y+1)) \\ &+ \lambda_R \beta_R \left[\alpha(p^o - d^o)(p^o - d^o + V_t(y) - V_t(y+1))\right] \\ &+ V_t(y+1) - V_t(y) \\ &= (V_t(y+1) - V_t(y))(1 - \lambda_P \beta_P(p^o) - \lambda_R \beta_R \alpha(p^o - d^o)) \\ &+ \lambda_P \beta_P(p^o) p^o + \left[\lambda_R \beta_R \alpha(p^o - d^o)(p^o - d^o)\right]. \end{aligned} \tag{A-11}$$

By the induction assumptions on (a) and (c), we have

$$V_t(y+1) - V_t(y) \geq V_{t-1}(y+1) - V_{t-1}(y) \geq V_{t-1}(y+2) - V_{t-1}(y+1).$$

Using this together with (A-11),we get

$$\begin{aligned} V_{t+1}(y+1) - V_t(y) &\geq (V_{t-1}(y+2) - V_{t-1}(y+1))(1 - \lambda_P \beta_P(p^o) - \lambda_R \beta_R \alpha(p^o - d^o)) \\ &+ \lambda_P \beta_P(p^o) p^o + \left[\lambda_R \beta_R \alpha(p^o - d^o)(p^o - d^o)\right]. \end{aligned} \tag{A-12}$$

From (A-10) and (A-12), we have

$$V_{t+1}(y+1) - V_t(y) \geq V_t(y+2) - V_{t-1}(y+1).$$

Also, using part (b) for $(y+1, t-1)$ (note that (ii) has already been verified for $y+t=k$), we have

$$V_t(y+1) - V_{t-1}(y+1) \geq V_{t+1}(y+1) - V_t(y+1).$$

Adding up both inequalities, it follows that

$$V_t(y+1) - V_t(y) \geq V_t(y+2) - V_t(y+1).$$

□

Lemma 6 *Let* $\Pi(p,\Delta) = a\Pi_1(p,\Delta) + b\Pi_2(p,\Delta)$ *where* $a \geq 0, b \geq 0$ *and* $\Pi_2(p,\Delta)$ *and* $\Pi_1(p,\Delta)$ *are as defined by (6) and (7). Let* $p_1^*(\Delta) = \inf\{p^* : \Pi_1(p^*,\Delta) \geq \Pi_1(p,\Delta), \forall p\}$, $p_2^*(\Delta) = \inf\{p^* : \Pi_2(p^*,\Delta) \geq \Pi_2(p,\Delta), \forall p\}$ *and* $p^*(\Delta) = \inf\{p^* : \Pi(p^*,\Delta) \geq \Pi(p,\Delta), \forall p\}$. *Then,* $p_1^*(\Delta)$, $p_2^*(\Delta)$ *and* $p^*(\Delta)$ *are all increasing in* Δ.

Proof of Lemma 6: We will first prove the result for $p_1^*(\Delta)$. Let $C := \{(p,\Delta) : \Delta > 0 \text{ and } p > \Delta\}$. By Theorem 8.1 on p.124 of Porteus (2002), it is sufficient to show that $\Pi_1(p,\Delta)$ is supermodular on C. Hence, we need to prove that the following inequality is true for any $x = (x_1, x_2) \in C$ and $y = (y_1, y_2) \in C$:

$$\Pi_1(x \wedge y) + \Pi_1(x \vee y) \geq \Pi_1(x) + \Pi_1(y) \tag{A-13}$$

where $x \wedge y = (\min(x_1, y_1), \min(x_2, y_2))$ and $x \vee y = (\max(x_1, y_1), \max(x_2, y_2))$. We will consider two cases:

Case 1, $x_1 \geq y_1$ and $x_2 \geq y_2$: In this case, the desired inequality holds trivially since $x \wedge y = y$ and $x \vee y = x$.

Case 2, $x_1 \geq y_1$ and $x_2 < y_2$: By substituting $\Pi_1(p,\Delta) = \beta_P(p)(p-\Delta)$ in (A-13) and after some algebra, one can show that the inequality in (A-13) is equivalent to the following:

$$\Pi_1(x \wedge y) + \Pi_1(x \vee y) - \Pi_1(x) - \Pi_1(y) = (y_2 - x_2)\left(\beta_P(y_1) - \beta_P(x_1)\right) \geq 0$$

The above inequality holds under the assumptions of Case 2 due to the fact that $\beta_P(x)$ is decreasing in x. This concludes the proof of the supermodularity of $\Pi_1(p,\Delta)$'s on C, which allows us to conclude $p_1^*(\Delta)$ is increasing in Δ.

As for $p_2^*(\Delta)$, the proof is similar, and uses the fact that $\Pi_2(p,\Delta)$ is supermodular. Finally, since $\Pi_1(p,\Delta)$ and $\Pi_2(p,\Delta)$ are supermodular, $\Pi(p,\Delta) = a\Pi_1(p,\Delta) + b\Pi_2(p,\Delta)$ is supermodular (by Lemma 8.3 on p.123 of Porteus (2002)), and this allows us to conclude that $p^*(\Delta)$ is increasing in Δ. □

Appendix B - Proofs for Section 4.2

Proposition 5 corresponds to parts (b), (d) and (e) of the following proposition and Theorem 2 corresponds to part (c).

Proposition 8 *Let $d_t^*(x,y)$ denote the largest optimal discount corresponding to the smallest optimal price when there are t time periods to go, there are x units of the regular product and y units of the promotional product. Suppose that $\delta_{11}+\delta_{22}<1$. Then,*
(a) $V_t(x+1,y)-V_t(x,y)\geq 0$, $V_t(x,y+1)-V_t(x,y)\geq 0$ for $x\geq 0$, $y\geq 0$, $t\geq 0$
(b) $\Delta(x+1,y+1,t)\geq\Delta(x,y+1,t)$ for $x\geq 0$, $y\geq 0$, $t\geq 1$
(c) $d_t^*(x,y)>0$ for $x\geq 0$, $y\geq 1$, $t\geq 1$.
(d) $\Delta(x,y+2,t)\leq\Delta(x,y+1,t)$ for $x\geq 0$, $y\geq 0$, $t\geq 1$
(e) $\Delta(x,y+1,t+1)\geq\Delta(x,y+1,t)$ for $x\geq 0$, $y\geq 0$, $t\geq 1$

Proof of Proposition 8: We skip the proof of part (a) since it immediately follows from (12) with a simple induction argument. Throughout the proof, recall the definition that

$$\Delta(x,y+1,t)=V_{t-1}(x,y+1)-V_{t-1}(x,y) \text{ for any } x,y\geq 0 \text{ and } t\geq 1.$$

In this proof, we utilize Lemma A1 of Netessine, Savin, and Xiao (2006), stated as Lemma 8 at the end of Appendix B.

Proof of (b): The proof is by induction. Note that (b) holds trivially when $t=1$ or $y=0$. Suppose that for some $t>1$,

$$\Delta(x+1,y+1,t)\geq\Delta(x,y+1,t) \tag{A-14}$$

We need to prove that

$$\Delta(x+1,y,t+1)\geq\Delta(x,y,t+1)$$

It is sufficient to consider two different cases.

Case 1: $x,y\geq 1$.

Since $\delta_{11}+\delta_{22}<1$, using (A-14) and Lemma 7, it follows that $d_t^*(x,y)>0$. Therefore, we can use (12) to write

$$\begin{aligned}
&V_t(x+1,y+1) - V_t(x,y+1) = \\
&\quad \lambda_R\beta_R \Big[V_{t-1}(x,y+1) - V_{t-1}(x-1,y+1) + \max_z \left\{\alpha(z)\left(z + V_{t-1}(x,y) - V_{t-1}(x,y+1)\right)\right\} \\
&\qquad - \max_z \left\{\alpha(z)\left(z + V_{t-1}(x-1,y) - V_{t-1}(x-1,y+1)\right)\right\}\Big] \\
&\quad + \lambda_P \Big[V_{t-1}(x+1,y+1) - V_{t-1}(x,y+1) + \max_z \left\{\beta_P(z)\left(z + V_{t-1}(x+1,y) - V_{t-1}(x+1,y+1)\right)\right\} \\
&\qquad - \max_z \left\{\beta_P(z)\left(z + V_{t-1}(x,y) - V_{t-1}(x,y+1)\right)\right\}\Big] \\
&\quad + (1 - \lambda_R\beta_R - \lambda_P)(V_{t-1}(x+1,y+1) - V_{t-1}(x,y+1)). \qquad \text{(A-15)}
\end{aligned}$$

Define

$$g(x) = \max_z\{\beta_P(z)(z-x)\} \text{ and } h(x) = \max_z\{\alpha(z)(z-x)\}.$$

Using the above definition, we can rewrite (A-15) as:

$$\begin{aligned}
V_t(x+1,y+1) - V_t(x,y+1) = \lambda_R\beta_R &\begin{bmatrix} & V_{t-1}(x,y+1) - V_{t-1}(x-1,y+1) \\ + & h(\Delta(x,y+1,t)) - h(\Delta(x-1,y+1,t)) \end{bmatrix} \\
+ \lambda_P &\begin{bmatrix} & V_{t-1}(x+1,y+1) - V_{t-1}(x,y+1) \\ + & g(\Delta(x+1,y+1,t)) - g(\Delta(x,y+1,t)) \end{bmatrix} \\
&+ (1 - \lambda_R\beta_R - \lambda_P)(V_{t-1}(x+1,y+1) - V_{t-1}(x,y+1))
\end{aligned}$$

It now follows from Lemma 8 and (A-14) that

$$\begin{aligned}
V_t(x+1,y+1) - V_t(x,y+1) \geq \lambda_R\beta_R &\begin{bmatrix} & V_{t-1}(x,y+1) - V_{t-1}(x-1,y+1) \\ - & \Delta(x,y+1,t) + \Delta(x-1,y+1,t) \end{bmatrix} \\
+ \lambda_P &\begin{bmatrix} & V_{t-1}(x+1,y+1) - V_{t-1}(x,y+1) \\ - & \Delta(x+1,y+1,t) + \Delta(x,y+1,t) \end{bmatrix} \\
&+ (1 - \lambda_R\beta_R - \lambda_P)(V_{t-1}(x+1,y) - V_{t-1}(x,y))
\end{aligned}$$

Using the definition of $\Delta(x,y,t)$, we can simplify the inequality above:

$$\begin{aligned}
V_t(x+1,y+1) - V_t(x,y+1) \geq\ &\lambda_R\beta_R(V_{t-1}(x,y) - V_{t-1}(x-1,y)) \\
&+ \lambda_P(V_{t-1}(x+1,y) - V_{t-1}(x,y)) \\
&+ (1 - \lambda_R\beta_R - \lambda_P)(V_{t-1}(x+1,y) - V_{t-1}(x,y))
\end{aligned}$$

Applying again Lemma 8 and (A-14), we can write

$$\begin{aligned}
V_t(x+1,y+1) - V_t(x,y+1) \geq{}& \lambda_R\beta_R \left[\begin{array}{cc} & V_{t-1}(x,y) - V_{t-1}(x-1,y) \\ + & h(\Delta(x,y,t)) - h(\Delta(x-1,y,t)) \end{array} \right] \\
&+ \lambda_P \left[\begin{array}{cc} & V_{t-1}(x+1,y) - V_{t-1}(x,y) \\ + & g(\Delta(x+1,y,t)) - g(\Delta(x,y,t)) \end{array} \right] \\
&+ (1-\lambda_R\beta_R - \lambda_P)(V_{t-1}(x+1,y) - V_{t-1}(x,y)) \\
={}& V_t(x+1,y) - V_t(x,y),
\end{aligned}$$

where the last equality follows from (12). Thus, we have shown that

$$V_t(x+1,y+1) - V_t(x,y+1) \geq V_t(x+1,y) - V_t(x,y),$$

which is equivalent to $\Delta(x+1,y,t+1) \geq \Delta(x,y,t+1)$.

Case 2: $x = 0, y \geq 1$.

As in Case 1, since $\delta_{11} + \delta_{22} < 1$, we have $d_t^*(x,y) > 0$, and we can write

$$\begin{aligned}
&V_t(1,y+1) - V_t(0,y+1) = \\
&\quad \lambda_R\beta_R \left[V_{t-1}(0,y+1) + \max_z \{\alpha(z)\,(z + V_{t-1}(0,y) - V_{t-1}(0,y+1))\} \right] \\
&\quad + \lambda_P \left[V_{t-1}(1,y+1) - V_{t-1}(0,y+1) + \max_z \{\beta_P(z)\,(z + V_{t-1}(1,y) - V_{t-1}(1,y+1))\} \right. \\
&\qquad \left. - \max_z \{\beta_P(z)\,(z + V_{t-1}(0,y) - V_{t-1}(0,y+1))\} \right] \\
&\quad + (1-\lambda_R\beta_R - \lambda_P)(V_{t-1}(1,y+1) - V_{t-1}(0,y+1)),
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
V_t(1,y+1) - V_t(0,y+1) ={}& \lambda_R\beta_R \left[V_{t-1}(0,y+1) + h(\Delta(0,y+1,t)) \right] \\
&+ \lambda_P \left[\begin{array}{cc} & V_{t-1}(1,y+1) - V_{t-1}(0,y+1) \\ + & g(\Delta(1,y+1,t)) - g(\Delta(0,y+1,t)) \end{array} \right] \\
&+ (1-\lambda_R\beta_R - \lambda_P)(V_{t-1}(1,y+1) - V_{t-1}(0,y+1))
\end{aligned}$$

Since $V_{t-1}(0,y+1) \geq V_{t-1}(0,y)$ (by Proposition 8(a)), $\Delta(1,y+1,t) \geq \Delta(0,y+1,t)$ (by (A-14)) and $\Delta(0,y+1,t) \leq \Delta(0,y,t)$ (by Proposition 3(b)), we obtain:

$$\begin{aligned}
V_t(1,y+1) - V_t(0,y+1) \geq{}& \lambda_R\beta_R \left[V_{t-1}(0,y) + h(\Delta(0,y,t)) \right] \\
&+ \lambda_P(V_t(1,y) - V_t(0,y)) \\
&+ (1-\lambda_R\beta_R - \lambda_P)(V_{t-1}(1,y+1) - V_{t-1}(0,y+1))
\end{aligned}$$

Since $\Delta(1,y,t) \geq \Delta(0,y,t)$ (by (A-14)), we can write

$$\begin{aligned} V_t(1,y+1) - V_t(0,y+1) \geq\ & \lambda_R\beta_R\left[V_{t-1}(0,y) + h(\Delta(0,y,t))\right] \\ & + \lambda_P\left[\begin{array}{cc} & V_{t-1}(1,y) - V_{t-1}(0,y) \\ + & g(\Delta(1,y,t)) - g(\Delta(0,y,t)) \end{array}\right] \\ & + (1-\lambda_R\beta_R-\lambda_P)(V_{t-1}(1,y+1) - V_{t-1}(0,y+1)) \\ =\ & V_t(1,y) - V_t(0,y), \end{aligned} \tag{A-16}$$

where the last equality follows from (12). Thus, we have shown that

$$V_t(1,y+1) - V_t(0,y+1) \geq V_t(1,y) - V_t(0,y),$$

which is equivalent to $\Delta(1,y+1,t+1) \geq \Delta(0,y+1,t+1)$, concluding the proof of (b).

Proof of (c): Part (c) follows from part (b) of the proposition and Lemma 7.

Proof of (d): The proof is by induction. Note that part (d) holds trivially when $t=1$. Furthermore, part (d) holds for $x=0$ by Proposition 3(b). Suppose that for some $t \geq 1$,

$$\Delta(x,y+2,t) \leq \Delta(x,y+1,t) \text{ for any } x \geq 1, y \geq 0 \tag{A-17}$$

We need to prove that

$$\Delta(x,y+2,t+1) \geq \Delta(x,y+1,t+1) \text{ for any } x \geq 1, y \geq 0$$

Part (c) of the proposition along with (12) allow us to write:

$$\begin{aligned} & \Delta(x,y+2,t+1) = V_t(x,y+2) - V_t(x,y+1) = \\ & \lambda_R\beta_R\left[\begin{array}{cl} & V_{t-1}(x-1,y+2) - V_{t-1}(x-1,y+1) \\ + & \max_z\left\{\alpha(z)\left(z + V_{t-1}(x-1,y+1) - V_{t-1}(x-1,y+2)\right)\right\} \\ - & \max_z\left\{\alpha(z)\left(z + V_{t-1}(x-1,y) - V_{t-1}(x-1,y+1)\right)\right\} \end{array}\right] \\ & + \lambda_P\left[\begin{array}{cl} & V_{t-1}(x,y+2) - V_{t-1}(x,y+1) \\ + & \max_z\left\{\beta_P(z)\left(z + V_{t-1}(x,y+1) - V_{t-1}(x,y+2)\right)\right\} \\ - & \max_z\left\{\beta_P(z)\left(z + V_{t-1}(x,y) - V_{t-1}(x,y+1)\right)\right\} \end{array}\right] \\ & + (1-\lambda_R\beta_R-\lambda_P)(V_{t-1}(x,y+2) - V_{t-1}(x,y+1)). \end{aligned} \tag{A-18}$$

Using the definitions of $h(x)$, $g(x)$ and $\Delta(x,y,t)$, we can simplify the equality above:

$$\begin{aligned} & \Delta(x,y+2,t+1) = \\ & \quad \lambda_R\beta_R\left[\Delta(x-1,y+2,t) + h(\Delta(x-1,y+2,t)) - h(\Delta(x-1,y+1,t))\right] \\ & \quad + \lambda_P\left[\Delta(x,y+2,t) + g(\Delta(x,y+2,t)) - g(\Delta(x,y+1,t))\right] \\ & \quad + (1-\lambda_R\beta_R-\lambda_P)\Delta(x,y+2,t). \end{aligned}$$

Using Lemma 8 and (A-17), we obtain from the above equality

$$\Delta(x, y+2, t+1) \leq \lambda_R\beta_R\Delta(x-1, y+1, t) + \lambda_P\Delta(x, y+1, t) + (1-\lambda_R\beta_R-\lambda_P)\Delta(x, y+2, t)$$

Since $\Delta(x-1, y+1, t) \leq \Delta(x-1, y, t)$, $\Delta(x, y+1, t) \leq \Delta(x, y, t)$ and $\Delta(x, y+2, t) \leq \Delta(x, y+1, t)$ (all by (A-17)), we can write

$$\begin{aligned}
&\Delta(x, y+2, t+1) \leq \\
&\quad \lambda_R\beta_R\left[\Delta(x-1, y+1, t) + h(\Delta(x-1, y+1, t)) - h(\Delta(x-1, y, t))\right] \\
&\quad + \lambda_P\left[\Delta(x, y+1, t) + g(\Delta(x, y+1, t)) - g(\Delta(x, y, t))\right] \\
&\quad + (1-\lambda_R\beta_R-\lambda_P)\Delta(x, y+1, t). \\
&\quad = V_t(x, y+1) - V_t(x, y) = \Delta(x, y+1, t+1),
\end{aligned}$$

where the next to last equality follows from (12). This concludes the proof of (d).

Proof of (e): Given the result proven in part (d) of the proposition, the proof of part (e) follows the same line of reasoning as the proof of Proposition 3(a). □

Before we state and prove Lemma 7, we first introduce new notation. For $x, y \geq 1$, we can rewrite (12) as

$$\begin{aligned}
V_t(x, y) &= (1-\lambda_R\beta_R)V_{t-1}(x, y) + \lambda_R\beta_R V_{t-1}(x-1, y) \\
&\quad + \max_{p\geq d\geq 0}\{\lambda_R\beta_R\alpha(p-d)(p-d-\Delta(x-1, y, t)) + \lambda_P\beta_P(p)(p-\Delta(x, y, t))\}
\end{aligned}$$

Let $\Pi_1(p, \Delta)$ and $\Pi_2(p, \Delta)$ be as defined by (6) and (7), that is:

$$\Pi_1(p, \Delta) = \beta_P(p)(p-\Delta) \text{ and } \Pi_2(p, \Delta) = \alpha(p)(p-\Delta).$$

Then, in period t, a firm with x units of regular product and y units of the promotional product is solving the following single-stage optimization problem:

$$\max_{p\geq d\geq 0}\{\lambda_P\Pi_1(p, \Delta(x, y, t)) + \lambda_R\beta_R\Pi_2(p-d, \Delta(x-1, y, t))\}.$$

Finally, define

$$z_1^*(\Delta) = \inf\{p^* : \Pi_1(p^*, \Delta) \geq \Pi_1(p, \Delta)\ \forall p\}$$

and

$$z_2^*(\Delta) = \inf\{p^* : \Pi_2(p^*, \Delta) \geq \Pi_2(p, \Delta)\ \forall p\}.$$

Lemma 7 *Suppose that for some fixed* $t > 0$*, we have*

$$\Delta(x+1, y+1, t) \geq \Delta(x, y+1, t) \tag{A-19}$$

for all $x, y \geq 0$*. Then, if* $\delta_{11} + \delta_{22} < 1$*, we have* $z_1^*(\Delta(x,y,t)) > z_2^*(\Delta(x-1,y,t))$ *and* $d_t^*(x,y) = z_1^*(\Delta(x,y,t)) - z_2^*(\Delta(x-1,y,t)) > 0$ *for all* $x \geq 1$ *and* $y \geq 0$*.*

Proof of Lemma 7: First, we note that $z_1^*(\Delta(x-1,y,t)) > z_2^*(\Delta(x-1,y,t))$. (The proof of this result follows as in Lemma 4 and is therefore omitted.) It now remains to show that $z_1^*(\Delta(x,y,t)) \geq z_1^*(\Delta(x-1,y,t))$. Given (A-19), it is easy to verify that

$$\frac{d}{dp}[\Pi_1(p, \Delta(x,y,t)) - \Pi_1(p, \Delta(x-1,y,t))] = \frac{d\beta_P(p)}{dp}(\Delta(x-1,y,t) - \Delta(x,y,t)) \geq 0. \tag{A-20}$$

From the definition of $z_1^*(\Delta(x-1,y,t))$,

$$\Pi_1(z_1^*(\Delta(x,y,t)), \Delta(x-1,y,t)) \leq \Pi_1(z_1^*(\Delta(x-1,y,t)), \Delta(x-1,y,t)).$$

Now, suppose for contradiction that $z_1^*(\Delta(x,y,t)) < z_1^*(\Delta(x-1,y,t))$. Then, from (A-20), it follows that

$$\Pi_1(z_1^*(\Delta(x,y,t)), \Delta(x,y,t)) \leq \Pi_1(z_1^*(\Delta(x-1,y,t)), \Delta(x,y,t)),$$

which is a contradiction to the definition of $z_1^*(\Delta(x,y,t))$. Hence, $z_1^*(\Delta(x,y,t)) \geq z_1^*(\Delta(x-1,y,t))$ and the result follows. □

Proof of Proposition 6: First, use Lemma 6 to observe that $z_1^*(\Delta)$ and $z_2^*(\Delta)$ are both increasing in Δ. Now, by Proposition 8(b) and Lemma 7, note that in period t with x units of regular inventory and y units of promotional inventory, the optimal announced price will be given by $z_1^*(\Delta(x,y,t))$ and the optimal upsell price will be given by $z_1^*(\Delta(x,y,t))$. The monotonicity results now follow from Proposition 8(b)–(d). □

Lemma 8 ***(Netessine, Savin, and Xiao)*** *Let* $g(x) = \max_{z \in A}\{\theta(z)(z-x)\}$ *where* $\theta(z)$ *is non-decreasing in* z *with* $0 \leq \theta(z) \leq 1$*. Then, for any* $x_1, x_2 \geq 0$*,*

$$\max(0, x_2 - x_1) \geq g(x_1) - g(x_2) \geq \min(0, x_2 - x_1).$$

Proof of Lemma 8: For a proof, see Netessine, Savin, and Xiao (2006). □

Appendix C - Model details and proofs for Section 4.3

We will require some changes to the notation defined earlier. The following is a list of the notation to be used in this section:

r_k : exogenously fixed price of regular product k

p_t : announced price of the promotional product in period t

d_{tk} : discount offered on the promotional product in the upsell stage to a customer who purchased regular product k in the initial stage of period t

λ_k : the probability that a potential customer for regular item k will arrive in a given period

λ_P : the probability that a potential customer for the promotional item will arrive in a given period

F_{ik} : the cdf of the reservation price of segment i customers for regular item k

F_{Pi} : the cdf of the reservation price of segment i customers for the promotional item

q_{ik} : the probability that a customer belongs to segment i for regular item k

q_{Pi} : the probability that a customer belongs to segment i for the promotional item

$\widehat{q}_{Pik}$: the probability that a customer belongs to segment i for the promotional item given that the customer bought regular item k in the initial stage

δ_{ijk} : the probability that a customer belongs to segment j for the promotional item given that she belongs to segment i for regular item k

β_k : the probability that a potential customer for regular product k will buy it at the exogenously-fixed price

$\beta_P(x)$: the probability that a potential customer for the promotional product will buy it at price x

$\alpha_k(x)$: given that a customer bought regular item k in the initial stage, the probability that the customer will buy the promotional item in the upsell stage at price x

Given the notation above, β_k, $\beta_P(x)$ and $\alpha_k(x)$ are obtained from q_{ik}, q_{Pi} and $\widehat{q}_{Pik}$ as before. The optimality equations, given by (5) for the single regular product case, can now be modified as follows:

$$\begin{aligned} V_t(y) &= V_{t-1}(y) + \max_{p_t, d_{kt}: p_t \geq d_{kt} \geq 0} \{\lambda_P \beta_P(p_t)(p_t + V_{t-1}(y-1) - V_{t-1}(y)) \\ &+ \left. \sum_{k=1}^{n} \lambda_k \beta_k \alpha_k (p_t - d_{tk})(p_t - d_{tk} + V_{t-1}(y-1) - V_{t-1}(y)) \right\}, \\ & y > 0, t = 1, \ldots, T \end{aligned} \quad \text{(A-21)}$$

Once again, define $\Delta(y,t) = V_{t-1}(y) - V_{t-1}(y-1)$ and

$$\begin{aligned} \Pi_1(p, \Delta) &= \beta_P(p)(p - \Delta), \\ \Pi_{2k}(p, \Delta) &= \alpha_k(p)(p - \Delta), k = 1, \ldots, n. \end{aligned}$$

Using the definitions above, we can write the optimality equations in (A-21) in the following alternative form:

$$\max_p \left\{ \lambda_P \Pi_1(p, \Delta(y,t)) + \sum_{k=1}^{n} \max_{0 \leq p_k \leq p} \{\lambda_k \beta_k \Pi_{2k}(p_k, \Delta(y,t))\} \right\} \quad \text{(A-22)}$$

As in the single regular product case, define $z_1^*(y,t)$ and $z_{2k}^*(t,y)$ as:

$$\begin{aligned} z_1^*(y,t) &= \inf\{p^* : \Pi_1(p^*, \Delta(y,t)) \geq \Pi_1(p, \Delta(y,t)), \forall p\} \\ z_{2k}^*(t,y) &= \inf\{p^* : \Pi_{2k}(p^*, \Delta(y,t)) \geq \Pi_{2k}(p, \Delta(y,t)), \forall p\} \end{aligned}$$

In addition, let $p_t^*(y)$ denote the optimal announced price of the promotional item with y units in inventory and t periods to go. (We pick the smallest maximizer when multiple maximizers exist.) It is given by

$$\begin{aligned} p_t^*(y) &= \inf \left\{ p^* : \lambda_P \Pi_1(p^*, \Delta(y,t)) + \sum_{k=1}^{n} \max_{0 \leq p_k \leq p^*} \{\lambda_k \beta_k \Pi_{2k}(p_k, \Delta(y,t))\} \right. \\ & \left. \geq \lambda_P \Pi_1(p, \Delta(y,t)) + \sum_{k=1}^{n} \max_{0 \leq p_k \leq p} \{\lambda_k \beta_k \Pi_{2k}(p_k, \Delta(y,t))\}, \forall p \right\} \end{aligned}$$

In order to prove Theorem 3, we will first prove a number of lemmas. The following lemma is analogous to Lemma 4, and it goes through as before:

Lemma 9 *For a given $k \in \{1, \ldots, n\}$, suppose that $\delta_{11k} + \delta_{22k} < (>)(=)1$. Then, $z_1^*(y,t) > (<)(=) z_{2k}^*(y,t)$.*

Next, we prove three new lemmas:

Lemma 10 *For any $i, j \in \{1, \ldots, n\}$, if $\widehat{q}_{P1i} - \widehat{q}_{P1j} > (<)(=) 0$, then $\frac{d(\Pi_{2i}(p,\Delta(y,t)) - \Pi_{2j}(p,\Delta(y,t)))}{dp} > (<)(=) 0$ for $p \in (\eta_2(y,t), \eta_1(y,t))$.*

Proof of Lemma 10: We have

$$\Pi_{2i}(p, \Delta(y,t)) - \Pi_{2j}(p, \Delta(y,t)) = (\alpha_i(p) - \alpha_j(p))(p - \Delta(y,t)).$$

Then,

$$\begin{aligned}
\frac{d(\Pi_{2i}(p, \Delta(y,t)) - \Pi_{2j}(p, \Delta(y,t)))}{dp} &= (\alpha_i(p) - \alpha_j(p)) + (\alpha_i'(p) - \alpha_j'(p))(p - \Delta(y,t)) \\
&= (\widehat{q}_{P1i} - \widehat{q}_{P1j})(\overline{F}_{P1}(p) - (p - \Delta(y,t)) f_{P1}(p)) \\
&+ (\widehat{q}_{P2i} - \widehat{q}_{P2j})(\overline{F}_{P2}(p) - (p - \Delta(y,t)) f_{P2}(p)) \\
&= (\widehat{q}_{P1i} - \widehat{q}_{P1j}) \\
&\times \left[\overline{F}_{P1}(p) - (p - \Delta(y,t)) f_{P1}(p) - \overline{F}_{P2}(p) + (p - \Delta(y,t)) f_{P2}(p)\right]
\end{aligned}$$

The result follows from the fact that the term in brackets above is strictly positive for $p \in (\eta_2(y,t), \eta_1(y,t))$ (by Lemma 1(b)). □

Lemma 11 *For any $i, j \in \{1, \ldots, n\}$, if $\widehat{q}_{P1i} - \widehat{q}_{P1j} > (<)(=) 0$, then $z_{2i}^*(y,t) > (<)(=) z_{2j}^*(y,t)$.*

Proof of Lemma 11: We will prove the result for the case where $\widehat{q}_{P1i} - \widehat{q}_{P1j} > 0$. The other cases follow similarly. Suppose, for a contradiction, that $\widehat{q}_{P1i} - \widehat{q}_{P1j} > 0$ and $z_{2i}^*(y,t) < z_{2j}^*(y,t)$. From the optimality of $z_{2i}^*(y,t)$ for $\Pi_{2i}(\cdot, \Delta(y,t))$, we have $\Pi_{2i}(z_{2i}^*(y,t), \Delta(y,t)) \geq \Pi_{2i}(z_{2j}^*(y,t), \Delta(y,t))$. Furthermore, from Lemma 10, we have $\Pi_{2i}(z_{2i}^*(y,t), \Delta(y,t)) - \Pi_{2j}(z_{2i}^*(y,t), \Delta(y,t)) < \Pi_{2i}(z_{2j}^*(y,t), \Delta(y,t)) - \Pi_{2j}(z_{2j}^*(y,t), \Delta(y,t))$. The last two inequalities together yield $\Pi_{2j}(z_{2i}^*(y,t), \Delta(y,t)) > \Pi_{2j}(z_{2j}^*(y,t), \Delta(y,t))$, which is a contradiction to the optimality of $z_{2j}^*(y,t)$ for $\Pi_{2j}(\cdot, \Delta(y,t))$. Hence, if $\widehat{q}_{P1i} - \widehat{q}_{P1j} > 0$, then we must have $z_{2i}^*(y,t) \geq z_{2j}^*(y,t)$. It remains to show that $z_{2i}^*(y,t) \neq z_{2j}^*(y,t)$.

Now, suppose for contradiction that $\widehat{q}_{P1i} - \widehat{q}_{P1j} > 0$ and $z_{2i}^*(y,t) = z_{2j}^*(y,t)$. From the optimality of $z_{2i}^*(y,t)$ for $\Pi_{2i}(\cdot, \Delta(y,t))$, we must have $\left.\frac{d\Pi_{2i}(p,\Delta(y,t))}{dp}\right|_{z_{2i}^*(y,t)} = 0$. Therefore, from Lemma 10, we have $\left.\frac{d\Pi_{2j}(p,\Delta(y,t))}{dp}\right|_{z_{2j}^*(y,t)} < 0$, which is a contradiction to the optimality of $z_{2j}^*(y,t)$ for $\Pi_{2j}(\cdot, \Delta(y,t))$. Hence, if $\widehat{q}_{P1i} - \widehat{q}_{P1j} > 0$, then we cannot have $z_{2i}^*(y,t) = z_{2j}^*(y,t)$. Therefore, it must be that $z_{2i}^*(y,t) > z_{2j}^*(y,t)$. □

Lemma 12 *Suppose the regular products are indexed so that* $\widehat{q}_{P11} \leq \ldots \leq \widehat{q}_{P1n}$. *Then:*
(a) $z^*_{21}(y,t) \leq \ldots \leq z^*_{2n}(y,t)$,
(b) $z^*_1(y,t) \leq p^*_t(y)$.

Proof of Lemma 12: Observe that, by Lemma 11 and our assumption that $\widehat{q}_{P11} \leq \ldots \leq \widehat{q}_{P1n}$, we have $z^*_{21}(y,t) \leq \ldots \leq z^*_{2n}(y,t)$. To see why $p^*_t(y) \geq z^*_1(y,t)$, note that $\max_{0 \leq p_k \leq p}\{\lambda_k \beta_k \Pi_{2k}(p_k, \Delta(y,t))\}$ is increasing in p. Hence, we notice from the optimality equations in (A-22) that we must have $p^*_t(y) \geq z^*_1(y,t)$. □

Proof of Theorem 3:

Proof of (a): First, $z^*_1(y,t) > z^*_{2k}(y,t)$ follows from Lemma 9. This along with $p^*_t(y) \geq z^*_1(y,t)$ (by Lemma 12(b)) yield $p^*_t(y) > z^*_{2k}(y,t)$. Therefore, when $p = p^*_t(y)$, setting $p_k = z^*_{2k}(y,t)$ is feasible for the optimization problem in (A-22), and it is optimal to do so since $z^*_{2k}(y,t)$ maximizes $\Pi_{2k}(p)$. The result follows.

Proof of (b): Since there is at least one dissimilar regular product, there exists $j \in \{1,\ldots,n\}$ such that $z^*_1(y,t) \leq z^*_{2j}(y,t)$ by Lemma 9. Then, it follows from Lemma 12(a) and (b) that there exists $m \in \{1,\ldots,n\}$ such that $p^*_t(y) > z^*_{2k}(y,t)$ for $k \leq m$. Therefore, as in part (a), when $p = p^*_t(y)$, setting $p_k = z^*_{2k}(y,t)$ for $k \leq m$ is feasible for the optimization problem in (A-22), and it is optimal to do so. The result follows. □