Operations Research
doi 10.1287/opre.1090.0701ec

e-companion
ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—"Personalized Dynamic Pricing of Limited Inventories" by Goker Aydin and Serhan Ziya, *Operations Research*, doi 10.1287/opre.1090.0701.

Online Supplement

“Personalized Dynamic Pricing ...” by Aydin and Ziya

Appendix A - Extension: Non-signaling Customers

Here we extend the model of Section 4 to consider the possibility that the firm may not be able to observe signals from all the customers. In the case where signals are volunteered by the customers themselves, some of the customers may simply choose to hide their signals for privacy concerns. For example, customers can choose to hide their identities online by simply deleting the cookies that track them. In the case where signals are typically directly observed by the firm without any need for customer volunteering, the signal may simply be unavailable for a group of customers. For instance, if the signal uses past purchasing data, the firm may not have sufficient information on relatively new customers.

We assume that the signal of a segment i customer is available with probability r_i.[5] At the beginning of each period, the firm announces a price that will be charged to customers who do not provide a signal. On the other hand, customers who provide signals may be offered a discount from the announced price. The amount of the discount, if offered, depends on the customer's signal. It is unlikely that a firm would announce a price and then charge a premium to customers who do provide a signal. Therefore, we assume that the firm never charges a signaling customer a price that is higher than the announced price, the price asked from the non-signaling customers.

Using the notation of Section 4, we can write the optimality equation as follows:

$$V_t^{NS}(y) = \max_p \left\{ \begin{array}{l} \lambda(q_1(1-r_1)\overline{F}_1(p) + q_2(1-r_2)\overline{F}_2(p))(p + V_{t-1}^{NS}(y-1)) \\ +\lambda(q_1(1-r_1)F_1(p) + q_2(1-r_2)F_2(p))V_{t-1}^{NS}(y) \\ +\lambda(q_1 r_1 + q_2 r_2)E_S\left[\max_{p_d \le p}\left\{ \begin{array}{l} (\widehat{q}_1(S)\overline{F}_1(p_d) + \widehat{q}_2(S)\overline{F}_2(p_d))(p_d + V_{t-1}^{NS}(y-1)) \\ +(\widehat{q}_1(S)F_1(p_d) + \widehat{q}_2(S)F_2(p_d))V_{t-1}^{NS}(y) \end{array}\right\}\right] \\ +(1-\lambda)V_{t-1}^{NS}(y) \end{array} \right\},$$

$$y > 0, t = 1, \ldots, T,$$

$$V_t^{NS}(0) = 0, t = 1, \ldots, T, \text{ and } V_0^{NS}(\cdot) = 0.$$

[5] An interesting extension would be to consider a model where customers act strategically and choose to volunteer their signal or not depending on the value of their signal. In such a model, it would be reasonable to assume that customers with lower signals would be more willing to provide their signals.

The optimality equation for $y > 0, t = 1, \ldots, T$ can more simply be written as

$$\begin{aligned} V_t^{NS}(y) =& V_{t-1}^{NS}(y) \\ &+ \lambda \left[\max_p \left\{ \begin{array}{l} (q_1(1-r_1)\overline{F}_1(p) + q_2(1-r_2)\overline{F}_2(p))(p - \Delta_t^{NS}(y)) \\ +(q_1 r_1 + q_2 r_2) E_S \left[\max_{p_d \le p} \left\{ (\widehat{q}_1(S)\overline{F}_1(p_d) + \widehat{q}_2(S)\overline{F}_2(p_d))(p_d - \Delta_t^{NS}(y)) \right\} \right] \end{array} \right\} \right] \end{aligned} \tag{A-1}$$

where

$$\Delta_t^{NS}(y) = V_{t-1}^{NS}(y) - V_{t-1}^{NS}(y-1)$$

The trade-off involved in determining p can be easily observed from (A-1). The firm would like to charge the price that will maximize the expected revenue from the customers who do not give signals but at the same time would like to make this price as high as possible so that it will have more flexibility in determining the personalized prices for the signaling customers.

Pricing decisions

Our objective here is to investigate how changes in the probabilities r_1 and r_2 will reflect on the optimal announced price and the discounts that the signaling customers will be offered. Let the optimal announced price for a given inventory level y and time t be denoted by p^*. Suppose that, at time t, there is a change in the signaling probabilities r_1 and r_2. The following proposition describes how the optimal price p^* is affected by such a change.

Proposition A-1 *The optimal announced price increases if:*
(a) the signaling probability for segment 2, r_2 increases, while the signaling probability for segment 1, r_1 remains the same, or
(b) the signaling probabilities are the same across both segments ($r_1 = r_2$) and they both increase by the same amount.

Proposition A-1 identifies scenarios in which the direction of change in the optimal price is known. Under the first scenario (Proposition A-1(a)), after the changes in signaling probabilities, a higher percentage of non-signaling customers are of segment 1, the segment with lower price sensitivity. Thus, the firm would like to charge the non-signaling customers a higher price. Since charging a higher announced price also increases the profits from signaling customers (as the firm will be more flexible in making personalized offers), the firm chooses to increase the announced price.

In the other scenario (Proposition A-1(b)), the signaling probabilities are the same for both segments before and after the increase in signaling probabilities. Here, the firm chooses to increase the announced price, because higher announced price is always good for the profit that will be obtained from signaling customers and a larger portion of the population now chooses to signal.

Proposition A-1 does not say anything about what happens when the percentage of signaling segment 1 customers increases while the percentage of signaling segment 2 customers remains the same. In such a case, the ratio of segment 1 customers among the non-signaling group decreases, which is an incentive for the firm to decrease the price. However, now, more customers provide a signal and, thus, there is also the incentive to increase the price so as to better price discriminate among signaling customers. Thus, it is not clear how the optimal price would change.

The firm's profits

In this section, our objective is to investigate the firm's profits under different scenarios each differing only in their signaling probabilities r_1 and r_2. We are mainly interested in identifying cases where the firm's profits are particularly large. With that objective, we have carried out a numerical study observing the effects of changes in the probabilities on the expected profits. Although we have considered several different examples (each assuming different distributions for the signals and different parameters for the reservation price distributions), the general characteristics of the relationship between the expected profits and the signaling probabilities did not appear to change from example to example. For brevity we report our observations on a single example.

In this example, signals are discrete random variables. Suppose that the probability mass function for S_1 is given by $g_1(1) = 0.2$, $g_1(2) = 0.3$, $g_1(3) = 0.5$ while the probability mass function for S_2 is given by $g_2(1) = 0.5$, $g_2(2) = 0.3$, and $g_2(3) = 0.2$. Also, suppose that $q_1 = 0.3$, $q_2 = 0.7$, $\lambda = 0.5$, F_1 is Weibull with shape and scale parameters, 2 and 100, respectively, and F_2 is Weibull with shape and scale parameters, 2 and 50. We compute the optimal expected profit for the initial inventory level of $y = 8$ and a sales horizon of $t = 24$ for different levels of signaling probabilities r_1 and r_2. Figure A-1 gives the plots of the optimal expected profits for each scenario. The data for the figure is provided in Table A-1.

What would the firm want its customers to do? Intuition might suggest that since signals help the firm make more informed decisions, the firm would want all the customers to signal (i.e., $r_1 = r_2 = 1$). However, the flaw with this argument is that it ignores the fact that the customers actually give another signal by simply not signaling. In fact, depending on the circumstances this other signal might be much more valuable to the firm than the signal they are hiding. Consider the case where $r_1 = 0$ and $r_2 = 1$. In other words, all segment 2 customers signal while all segment

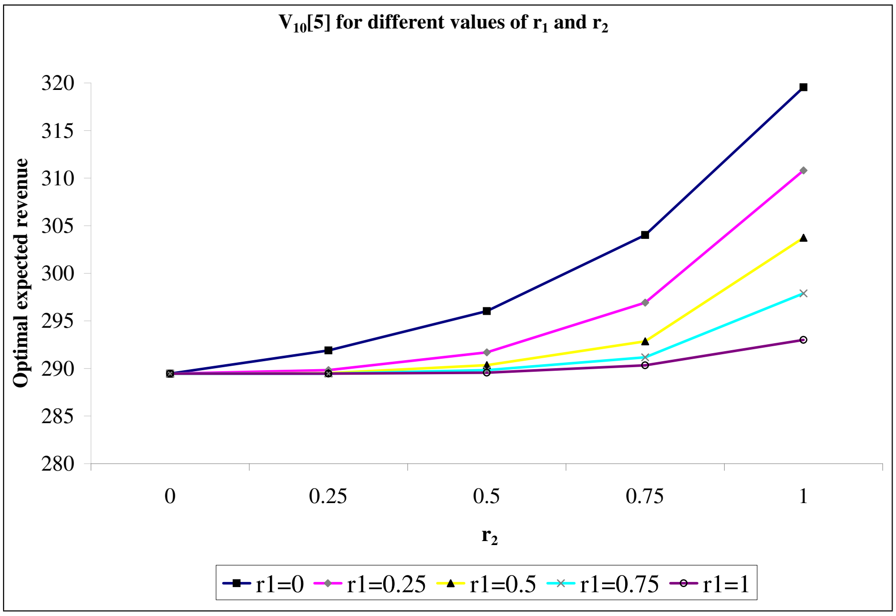

Figure A-1: Plot of optimal expected revenues with an initial inventory level of 8 and a sales horizon of 24 under different signaling probabilities. The x-axis is for r_2. Each curve corresponds to the optimal expected revenue for different values of r_1, indicated by different types of markers.

	$\mathbf{r_2 = 0}$	$\mathbf{r_2 = 0.25}$	$\mathbf{r_2 = 0.5}$	$\mathbf{r_2 = 0.75}$	$\mathbf{r_2 = 1}$
$\mathbf{r_1 = 0}$	289.462	291.893	296.021	304.021	319.540
$\mathbf{r_1 = 0.25}$	289.462	289.832	291.697	296.918	310.818
$\mathbf{r_1 = 0.5}$	289.462	289.504	290.343	292.860	303.730
$\mathbf{r_1 = 0.75}$	289.462	289.462	289.826	291.177	297.893
$\mathbf{r_1 = 1.0}$	289.462	289.462	289.554	290.340	293.001

Table A-1: Optimal expected revenues with an initial inventory level of 8 and a sales horizon of 24 under different signaling probabilities.

1 customers hide their signal. This is the perfect scenario for the firm since it knows the segment identity of every customer and therefore can price discriminate across the two segments. In this case, the signals that the customers voluntarily give are in fact useless since just the fact that they signal reveals all the information that the firm can possibly know anyway. Now, suppose that more customers signal. That means all segment 2 customers plus some of the segment 1 customers are signaling. In this case, if a customer does not signal the firm still knows that she is of segment 1, but how about a signaling customer? Now, the message that the firm is getting from the signaling customers is blurred as the firm no longer knows with certainty what segment a signaling customer

belongs to. Therefore, there is not a perfect price discrimination opportunity here and the expected profits are lower. This can be observed from Figure 1.

Now, suppose that $r_1 = 1$ and $r_2 = 0$ so that all segment 1 customers signal but none of the segment 2 customers does so. Then, again, the firm can perfectly identify the segment identities of the customers. However, in this case, the optimal expected profit is not as large (in fact it is the scenario with the lowest profits along with other scenarios for which $r_2 = 0$) since the firm cannot price discriminate due to the fact that signaling customers cannot be charged prices that are higher than the price charged to non-signaling customers. Hence, the firm has perfect information but it cannot make any use of it.

The firm would then ideally want all segment 2 customers to signal while all segment 1 customers not to signal. As we can see from Figure 1, the farther away we are from this ideal scenario, the lower are the optimal expected profits. More specifically, profits decrease with r_1 for any fixed value of r_2 and increase with r_2 for any fixed value of r_1.

Appendix B - Definitions of Stochastic Orders Used in the Paper

The following definitions are based on Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). Here, we make slight changes such as restricting the definitions to non-negative random variables having the same support.

Suppose that X and Y are two non-negative random variables having the same support with corresponding cumulative distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, respectively.

Definition A-1 ***Usual Stochastic Ordering:*** *Suppose that $F_X(x) \leq F_Y(x)$ for all $x \in (0, \infty)$. Then F_X is said to be greater than F_Y in the usual stochastic order (denoted by $F_X \geq_{st} F_Y$).*

Definition A-2 ***Failure Rate Ordering:*** *Suppose that $F_X(\cdot)$ and $F_Y(\cdot)$ are absolutely continuous with failure rate functions $r_X(\cdot)$ and $r_Y(\cdot)$, respectively. If $r_X(x) \leq r_Y(x)$ (or equivalently $\frac{1-F_X(x)}{1-F_Y(x)}$ is increasing) over the common support of X and Y, then we say that F_X is greater than F_Y in failure rate ordering (denoted by $F_X \geq_{fr} F_Y$).*

Suppose that X and Y are discrete random variables that take values in the set $\mathcal{M}$. Then, we say that F_X is greater than F_Y in failure rate ordering (denoted by $F_X \geq_{fr} F_Y$) if

$$\frac{P\{X = m\}}{P\{X \geq m\}} \leq \frac{P\{Y = m\}}{P\{Y \geq m\}}$$

for all $m \in \mathcal{M}$.

Definition A-3 ***Likelihood Ratio Ordering:*** *Suppose that the following condition holds:*

$$P\{X \in A\}P\{Y \in B\} \leq P\{X \in B\}P\{Y \in A\} \tag{A-2}$$

for all measurable sets A and B in $\mathbb{R}^+$ such that $A \leq B$, where $A \leq B$ means that $x \in A$ and $y \in B$ implies that $x \leq y$. Then, F_X is said to be greater than F_Y in the likelihood ratio ordering (denoted by $F_X \geq_{lr} F_Y$).

If X and Y are continuous random variables, and $f_X(\cdot)$ and $f_Y(\cdot)$ are the corresponding probability density functions, Condition (A-2) *is equivalent to $\frac{f_X(x)}{f_Y(x)}$ being increasing over the common support of X and Y.*

If X and Y are discrete random variables, and $f_X(\cdot)$ and $f_Y(\cdot)$ are the corresponding probability mass functions, Condition (A-2) *is equivalent to $\frac{f_X(x)}{f_Y(x)}$ being increasing over the common support of X and Y.*

Appendix C - Dynamic Programming Formulation for Section 5.2

In order to obtain an equivalent, but more tractable formulation of the optimality equations given by (5), we introduce $\widetilde{q}_{ij}(z)$, the probability that a customer is from segment i, given that she has been classified as a class-j customer, $i, j = 1, 2$. For example, given threshold signal z, $\widetilde{q}_{12}(z)$ is the probability that a customer who provided a signal less than z (and, thus, were classified as a class-2 customer) is in fact from segment-1. Using Bayes' rule, $\widetilde{q}_{ij}(z)$ is given by

$$\begin{aligned}\widetilde{q}_{i1}(z) &= \frac{q_i\overline{G}_i(z)}{q_1\overline{G}_1(z) + q_2\overline{G}_2(z)},\ i = 1, 2,\\ \widetilde{q}_{i2}(z) &= \frac{q_i G_i(z)}{q_1 G_1(z) + q_2 G_2(z)},\ i = 1, 2.\end{aligned} \tag{A-3}$$

Furthermore, let

$$\beta_j(z, p) := \widetilde{q}_{1j}(z)\overline{F}_1(p) + \widetilde{q}_{2j}(z)\overline{F}_2(p). \tag{A-4}$$

Note that $\beta_j(z, p)$ is the probability that a customer will buy the product at price p, given that the customer has been classified as class-j when the threshold signal is z. Using this notation one can show that $V_t^{FT}(y, z)$ can be written as follows:

$$\begin{aligned}V_t^{FT}(y, z) &= V_{t-1}^{FT}(y, z) + \lambda\left(q_1\overline{G}_1(z) + q_2\overline{G}_2(z)\right)\max_{p_1}\left\{\beta_1(z, p_1)\left(p_1 - \Delta_t^{FT}(y, z)\right)\right\}\\ &+ \lambda\left(q_1 G_1(z) + q_2 G_2(z)\right)\max_{p_2}\left\{\beta_2(z, p_2)\left(p_2 - \Delta_t^{FT}(y, z)\right)\right\}, y > 0, t = 1, \ldots, T\end{aligned}$$

where

$$\Delta_t^{FT}(y, z) = V_{t-1}^{FT}(y, z) - V_{t-1}^{FT}(y - 1, z), y > 0, t = 1, \ldots, T. \tag{A-5}$$

In the optimality equation above, the first maximization corresponds to choosing the price to be charged to a class-1 customer and the latter maximization to choosing the price for a class-2 customer.

Appendix D - Proofs of the Results

Proof of Theorem 1 and Proposition 1: For the purposes of this proof, define $\widehat{\Pi}(x, p, \Delta) := \alpha(x, p)(p - \Delta)$. Let $\widehat{p}(x, \Delta)$ denote the smallest optimizer of $\widehat{\Pi}(x, p, \Delta)$. Using this definition, note that $p^*(x, y, t)$ is given by $\widehat{p}(x, \Delta_t(y))$. In addition, define $\overline{\Pi}_i(p, \Delta) := (p - \Delta)\overline{F}_i(p)$, $i = 1, 2$. Let $\overline{p}_i(\Delta)$ denote the optimizer of $\overline{\Pi}_i(p, \Delta)$.

We now apply Lemma A-1 to prove Theorem 1. By Lemma A-1(a), $\widehat{p}(x, \Delta_t(y)) \in [\overline{p}_2(\Delta_t(y)), \overline{p}_1(\Delta_t(y))]$. By Lemma A-1(b), $\Pi(x, p, \Delta_t(y))$ is supermodular in p and x for $p \in [\overline{p}_2(\Delta_t(y)), \overline{p}_1(\Delta_t(y))]$ and $x \geq 0$. Therefore, $\widehat{p}(x, \Delta_t(y))$ is increasing in x, which concludes the proof of Theorem 1.

Proposition 1(a) follows from Lemma A-1(a). The rest of the proposition follows directly from part (a) of the proposition. □

Proof of Theorem 2: Given time t, inventory level y, and the customer signal x, it is optimal to charge p_1 if

$$\alpha(x, p_1)\,(p_1 - \Delta_t(y)) \geq \alpha(x, p_2)\,(p_2 - \Delta_t(y))$$

where $\Delta_t(y)$ is the marginal value of the inventory. This condition can be equivalently written as

$$\Delta_t(y) \geq \frac{p_2\alpha(x, p_2) - p_1\alpha(x, p_1)}{\alpha(x, p_2) - \alpha(x, p_1)}.$$

Now let

$$A(x) = \frac{p_2\alpha(x, p_2) - p_1\alpha(x, p_1)}{\alpha(x, p_2) - \alpha(x, p_1)}.$$

Then, it is sufficient to show that $A(x_1) \leq A(x_2)$ for $x_1 > x_2$. After some algebra, one can show that $A(x_1) - A(x_2)$ is equivalent in sign to

$$(p_1 - p_2)\,(\alpha(x_1, p_2)\alpha(x_2, p_1) - \alpha(x_1, p_1)\alpha(x_2, p_2))\,.$$

Since $p_1 > p_2$, the sign of $B(x) := \alpha(x_1, p_2)\alpha(x_2, p_1) - \alpha(x_1, p_1)\alpha(x_2, p_2)$ determines the sign of $A(x_1) - A(x_2)$. Substituting for $\alpha(x, p)$ from (2), we find:

$$B(x) = -\left(\overline{F}_1(p_1)\overline{F}_2(p_2) - \overline{F}_1(p_2)\overline{F}_2(p_1)\right)(\widehat{q}_1(x_1)\widehat{q}_2(x_2) - \widehat{q}_1(x_2)\widehat{q}_2(x_1))\,.$$

Now, $\overline{F}_1(p_1)\overline{F}_2(p_2) - \overline{F}_1(p_2)\overline{F}_2(p_1) \geq 0$ follows from the facts that $F_1 \geq_{fr} F_2$ and the definition of failure rate ordering given in Appendix B. Therefore, we will conclude the proof if we can show that $\widehat{q}_1(x_1)\widehat{q}_2(x_2) - \widehat{q}_1(x_2)\widehat{q}_2(x_1)$ is positive. Substituting for $\widehat{q}_i(x)$ from (1) and after some algebra, one can verify that $\widehat{q}_1(x_1)\widehat{q}_2(x_2) - \widehat{q}_1(x_2)\widehat{q}_2(x_1)$ is equivalent in sign to

$$q_1 q_2 \left(g_1(x_1)g_2(x_2) - g_1(x_2)g_2(x_1)\right)$$

The term in the parenthesis above is positive, due to the assumption that $G_1 \geq_{lr} G_2$ and the definition of likelihood ratio ordering (see Appendix B). This concludes the proof. □

Proof of Proposition 2: Note that $z^*(y,t)$ is given by the (smallest) value of z that maximizes the function $\Pi(z, p_1, p_2, \Delta_t^{FP}(y))$ defined in the statement of Lemma A-2. By Lemma A-2, $\Pi(z, p_1, p_2, \Delta_t^{FP}(y))$ is submodular in z and $\Delta_t^{FP}(y)$. Therefore, $z^*(y,t)$ is decreasing in $\Delta_t^{FP}(y)$. The result now follows from the fact that $\Delta_t^{FP}(y)$ is decreasing in y and increasing in t (by Lemma A-5). □

Proof of Proposition 3: Here, we make use of the formulation provided in Appendix C. For the purposes of this proof, define $\widetilde{\Pi}_j(z,p,\Delta) := \beta_j(z,p)(p-\Delta)$, where $\beta_j(z,p)$ is as defined by (A-4) in Appendix C. Let $\widetilde{p}_j(z,\Delta)$ denote the smallest optimizer of $\widetilde{\Pi}_j(z,p,\Delta)$. Note that, with this definition, $p_j^*(z,y,t)$ is given by $\widetilde{p}_j(z, \Delta_t^{FT}(y,z))$. In addition, define $\overline{\Pi}_i(p,\Delta) := (p-\Delta)\overline{F}_i(p)$, $i = 1,2$. Let $\overline{p}_i(\Delta)$ denote the optimizer of $\overline{\Pi}_i(p,\Delta)$. We can now utilize Lemma A-3 to prove the proposition.

The proof of Proposition 3(a) is by contradiction. Suppose $\widetilde{p}_1(z, \Delta_t^{FT}(y,z)) < \widetilde{p}_2(z, \Delta_t^{FT}(y,z))$. For the sake of exposition, we suppress the arguments of $\widetilde{p}_1$ and $\widetilde{p}_2$ in what follows. By definition of $\widetilde{p}_j$, we have

$$\widetilde{\Pi}_2(z, \widetilde{p}_2, \Delta_t^{FT}(y,z)) > \widetilde{\Pi}_2(z, \widetilde{p}_1, \Delta_t^{FT}(y,z)).$$

In addition, due to the assumption that $\widetilde{p}_1 < \widetilde{p}_2$, it follows from Lemma A-3(a),(b) that

$$\widetilde{\Pi}_1(z, \widetilde{p}_2, \Delta_t^{FT}(y,z)) - \widetilde{\Pi}_2(z, \widetilde{p}_2, \Delta_t^{FT}(y,z)) \geq \widetilde{\Pi}_1(z, \widetilde{p}_1, \Delta_t^{FT}(y,z)) - \widetilde{\Pi}_2(z, \widetilde{p}_1, \Delta_t^{FT}(y,z)).$$

The last two inequalities together imply that

$$\widetilde{\Pi}_1(z, \widetilde{p}_2, \Delta_t^{FT}(y,z)) > \widetilde{\Pi}_1(z, \widetilde{p}_1, \Delta_t^{FT}(y,z)),$$

which yields a contradiction to the optimality of $\widetilde{p}_1$ for $\widetilde{\Pi}_1$, thus concluding the proof.

We next prove Proposition 3(b). By Lemma A-3(a), $\widetilde{p}_j(z, \Delta_t^{FT}(y,z)) \in [\overline{p}_2(\Delta_t^{FT}(y,z)), \overline{p}_1(\Delta_t^{FT}(y,z))]$. By Lemma A-3(c), for a given $\Delta > 0$, $\widetilde{\Pi}_j(z,p,\Delta)$ is supermodular in p and z for $p \in [\overline{p}_2(\Delta), \overline{p}_1(\Delta)]$

and $z \geq 0$. Note that when z changes in period t only, $\Delta_t^{FT}(y,z)$ is not affected (since $\Delta_t^{FT}(y,z) = V_{t-1}^{FT}(y,z) - V_{t-1}^{FT}(y-1,z)$). Therefore, we can apply Lemma A-3(c) to conclude that $\widetilde{p}_j(z, \Delta_t^{FT}(y,z))$ increases if z increases in period t only. □

Proof of Proposition A-1:

Proof of (a): To simplify the presentation, we first define the following two functions:

$$\Gamma_1(p, r_2) = \left(q_1(1-r_1)\overline{F}_1(p) + q_2(1-r_2)\overline{F}_2(p)\right)(p - \Delta)$$

and

$$\Gamma_2(p, r_2) = (q_1 r_1 + q_2 r_2)\Theta(p)$$

where

$$\Theta(p) = E_S\left[\max_{p_d \leq p}\left\{(\widehat{q}_1(S)\overline{F}_1(p_d) + \widehat{q}_2(S)\overline{F}_2(p_d))(p_d - \Delta)\right\}\right].$$

and $\Delta = \Delta_t^{NS}(y)$. Now, for a given r_2, notice that the optimization problem in (A-1) can be written as

$$\max_p\{\Gamma_1(p, r_2) + \Gamma_2(p, r_2)\}.$$

For any value of r_2, using arguments similar to those in Lemma A-1(a), it can be shown that the optimal value of p resides in the interval $[\bar{p}_2(\Delta), \bar{p}_1(\Delta)]$. We will next show that both $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ are supermodular in p and r_2 for $p \in [\bar{p}_2(\Delta), \bar{p}_1(\Delta)]$. That will be sufficient to conclude that the optimal value of p is non-decreasing in r_2, since the summation of two supermodular functions is also supermodular.

To show the supermodularity of $\Gamma_1(p, r_2)$, we note that

$$\frac{\partial^2 \Gamma_1(p, r_2)}{\partial p \partial r_2} = q_2\left(f_2(p)(p - \Delta) - (1 - F_2(p))\right) \geq 0$$

where the inequality follows from the fact that $p \geq \bar{p}_2(\Delta)$. Thus, $\Gamma_1(p, r_2)$ is supermodular for $p \in [\bar{p}_2(\Delta), \bar{p}_1(\Delta)]$.

Now, in order to establish the supermodularity of $\Gamma_2(p, r_2)$, it is sufficient to show that $\Gamma_2(p, r_2)$ has increasing differences. Let $\bar{p}_1(\Delta) \geq p_1 \geq p_2 \geq \bar{p}_2(\Delta)$ and $1 \geq r_2^1 \geq r_2^2 \geq 0$. Then, we need to show that

$$\Gamma_2(p_1, r_2^1) - \Gamma_2(p_1, r_2^2) \geq \Gamma_2(p_2, r_2^1) - \Gamma_2(p_2, r_2^2).$$

After some algebra, one can show that the above inequality is equivalent to $(r_2^2 - r_2^1)(\Theta(p_2) - \Theta(p_1)) \geq 0$, which holds because $\Theta(p_1) \geq \Theta(p_2)$ and $r_2^1 \geq r_2^2$. Hence, $\Gamma_2(p, r_2)$ is supermodular, concluding the proof.

Proof of (b): The optimization problem in (A-1) can be written as

$$V_t(y) = V_{t-1}(y) + \lambda \max_p \{(1-r)\Lambda_1(p) + r\Lambda_2(p)\}$$

where

$$\Lambda_1(p) = (q_1 \overline{F}_1(p) + q_2 \overline{F}_2(p))(p - \Delta)$$

and

$$\Lambda_2(p) = E_S \left[\max_{p_d \le p} \left\{ (\widehat{q}_1(S)\overline{F}_1(p_d) + \widehat{q}_2(S)\overline{F}_2(p_d))(p_d - \Delta) \right\} \right].$$

Consider two scenarios: One in which $r_1 = r_2 = r$ with the corresponding optimal price p^* (or the smallest optimizer if there is more than one) and the other in which $r_1 = r_2 = \widehat{r} > r$ with the corresponding optimal price $\widehat{p}^*$ (again the smallest optimizer if there is more than one). Suppose for contradiction that $\widehat{r} > r$ but $p^* > \widehat{p}^*$. Then,

$$\Lambda_2(p^*) \ge \Lambda_2(\widehat{p}^*). \tag{A-6}$$

By optimality of p^* for $r_1 = r_2 = r$, we have

$$(1-r)\Lambda_1(p^*) + r\Lambda_2(p^*) > (1-r)\Lambda_1(\widehat{p}^*) + r\Lambda_2(\widehat{p}^*),$$

which can also be written as

$$\Lambda_1(p^*) - \Lambda_1(\widehat{p}^*) > \frac{r}{1-r}(\Lambda_2(\widehat{p}^*) - \Lambda_2(p^*)). \tag{A-7}$$

In addition, by optimality of $\widehat{p}^*$ for $r_1 = r_2 = \widehat{r}$, we have

$$(1-\widehat{r})\Lambda_1(p^*) + \widehat{r}\Lambda_2(p^*) \le (1-\widehat{r})\Lambda_1(\widehat{p}^*) + \widehat{r}\Lambda_2(\widehat{p}^*),$$

which can also be written as

$$\Lambda_1(p^*) - \Lambda_1(\widehat{p}^*) \le \frac{\widehat{r}}{1-\widehat{r}}(\Lambda_2(\widehat{p}^*) - \Lambda_2(p^*)). \tag{A-8}$$

But, (A-7) together with (A-8) is a contradiction to (A-6). Thus, we must have $p^* \le \widehat{p}^*$. □

Lemma A-1 *Define $\widehat{\Pi}(x,p,\Delta) := \alpha(x,p)(p-\Delta)$. Let $\widehat{p}(x,\Delta)$ denote the smallest optimizer of $\widehat{\Pi}(x,p,\Delta)$. Define $\overline{\Pi}_i(p,\Delta) := (p-\Delta)\overline{F}_i(p)$, $i = 1,2$. Let $\overline{p}_i(\Delta)$ denote the optimizer of $\overline{\Pi}_i(p,\Delta)$. Then:*

(a) (Aydin and Ziya, 2008) $\overline{p}_1(\Delta) \ge \widehat{p}(x,\Delta) \ge \overline{p}_2(\Delta)$.

(b) At a fixed $\Delta \ge 0$, $\widehat{\Pi}(x,p,\Delta)$ is supermodular in x and p for $x \ge 0$ and $p \in [\overline{p}_2(\Delta), \overline{p}_1(\Delta)]$.

Proof of Lemma A-1:

Proof of (a): The proof of this part is similar to Lemma 1 in the online supplement to Aydin and Ziya (2008). We include the proof here for the sake of completeness. It is not difficult to show that $\overline{\Pi}_i(p,\Delta)$ is strictly unimodal in p due to assumption $(A3)$. (See, for example, Lariviere and Porteus, 2001.) Since $\overline{\Pi}_i(p,\Delta)$ is strictly unimodal in p, $\widehat{p}_i(\Delta)$ must satisfy the following first-order condition (FOC):

$$\overline{F}_i(p) - (p-\Delta)f_i(p) = \overline{F}_i(p)\left(1-(p-\Delta)\frac{f_i(p)}{\overline{F}_i(p)}\right) = 0, i = 1,2. \tag{A-9}$$

Now the FOCs in (A-9) along with assumption $(A4)$ imply that $\overline{p}_1(\Delta) \geq \overline{p}_2(\Delta)$. To prove that $\widehat{p}(x,\Delta) \in [\overline{p}_2(\Delta), \overline{p}_1(\Delta)]$, we first observe that $\widehat{\Pi}(x,p,\Delta)$ can be written as

$$\widehat{\Pi}(x,p,\Delta) = \widehat{q}_1(x)\overline{\Pi}_1(p,\Delta) + \widehat{q}_2(x)\overline{\Pi}_2(p,\Delta)$$

Now, since $\overline{\Pi}_i(p,\Delta)$, $i=1,2$ are unimodal in p and $\overline{p}_1(\Delta) \geq \overline{p}_2(\Delta)$, it follows that $\widehat{\Pi}(x,p,\Delta)$ is increasing in p for $p \leq \overline{p}_2(\Delta)$ and decreasing in p for $p \geq \overline{p}_1(\Delta)$. Therefore, the optimizer of $\widehat{\Pi}(x,p,\Delta)$, denoted by $\widehat{p}(x,\Delta)$, must be in $[\overline{p}_2(\Delta), \overline{p}_1(\Delta)]$.

Proof of (b): First, note that $\widehat{\Pi}(x,p,\Delta)$ being supermodular in x and p is equivalent to $\widehat{\Pi}(x,p,\Delta)$ having increasing differences in x and p. Therefore, we will prove that, for $x_1 > x_2$, the difference $\widehat{\Pi}(x_1,p,\Delta) - \widehat{\Pi}(x_2,p,\Delta)$ is increasing in p.

$$\frac{\partial\left(\widehat{\Pi}(x_1,p,\Delta)-\widehat{\Pi}(x_2,p,\Delta)\right)}{\partial p} = (p-\Delta)\left(\frac{\partial\alpha(x_1,p)}{\partial p} - \frac{\partial\alpha(x_2,p)}{\partial p}\right) + \alpha(x_1,p) - \alpha(x_2,p) \tag{A-10}$$

where

$$\alpha(x,p) = \widehat{q}_1(x)\overline{F}_1(p) + \widehat{q}_2(x)\overline{F}_2(p) \tag{A-11}$$

$$\frac{\partial\alpha(x,p)}{\partial p} = -\widehat{q}_1(x)f_1(p) - \widehat{q}_2(x)f_2(p) \tag{A-12}$$

Substituting from (A-11) and (A-12) in (A-10) and rearranging the terms, we obtain:

$$\begin{aligned}\frac{\partial\left(\widehat{\Pi}(x_1,p,\Delta)-\widehat{\Pi}(x_2,p,\Delta)\right)}{\partial p} &= \left[-(p-\Delta)f_1(p)+\overline{F}_1(p)\right](\widehat{q}_1(x_1)-\widehat{q}_1(x_2)) \\ &+ \left[-(p-\Delta)f_2(p)+\overline{F}_2(p)\right](\widehat{q}_2(x_1)-\widehat{q}_2(x_2))\end{aligned}$$

Because $\widehat{q}_1(x) + \widehat{q}_2(x) = 1$ for any x, we have $\widehat{q}_1(x_1) - \widehat{q}_1(x_2) = -\widehat{q}_2(x_1) + \widehat{q}_2(x_2)$. Using this observation, we can write the above equality as:

$$\begin{aligned}\frac{\partial\left(\widehat{\Pi}(x_1,p,\Delta)-\widehat{\Pi}(x_2,p,\Delta)\right)}{\partial p} &= (\widehat{q}_1(x_1)-\widehat{q}_1(x_2)) \\ &\times \left\{\left[-(p-\Delta)f_1(p)+\overline{F}_1(p)\right] - \left[-(p-\Delta)f_2(p)+\overline{F}_2(p)\right]\right\}\end{aligned} \tag{A-13}$$

Observe that

$$\overline{F}_1(p) - (p-\Delta) f_1(p) = \frac{\partial \overline{\Pi}_1(p,\Delta)}{\partial p} \geq 0 \text{ for } p \leq \overline{p}_1(\Delta),$$

and

$$\overline{F}_2(p) - (p-\Delta) f_2(p) = \frac{\partial \overline{\Pi}_2(p,\Delta)}{\partial p} \leq 0 \text{ for } p \geq \overline{p}_2(\Delta).$$

Therefore, the term in curly brackets in (A-13) is positive, and we will conclude the proof if we can show that $\widehat{q}_1(x_1) - \widehat{q}_1(x_2) \geq 0$. Now:

$$\widehat{q}_1(x_1) - \widehat{q}_1(x_2) = \frac{q_1 q_2 \left(g_1(x_1) g_2(x_2) - g_1(x_2) g_2(x_1)\right)}{\left(q_1 g_1(x_1) + q_2 g_2(x_1)\right)\left(q_1 g_1(x_2) + q_2 g_2(x_2)\right)} \geq 0$$

where the inequality follows from the fact that $\frac{g_1(x_1)}{g_2(x_1)} \geq \frac{g_1(x_2)}{g_2(x_2)}$, because $G_1(\cdot) \geq_{lr} G_2(\cdot)$ by assumption $(A5)$. □

Lemma A-2 *Given $p_1 > p_2$, define*

$$\begin{aligned} \Pi(z, p_1, p_2, \Delta) \quad &:= \quad \left(q_1 \overline{G}_1(z) \overline{F}_1(p_1) + q_2 \overline{G}_2(z) \overline{F}_2(p_1)\right)(p_1 - \Delta) \\ &+ \quad \left(q_1 G_1(z) \overline{F}_1(p_2) + q_2 G_2(z) \overline{F}_2(p_2)\right)(p_2 - \Delta). \end{aligned}$$

Then, $\Pi(z, p_1, p_2, \Delta)$ is submodular in Δ and z for $\Delta \geq 0$ and $z \geq 0$.

Proof of Lemma A-2: First, note that $\Pi(z, p_1, p_2, \Delta)$ being submodular in z and Δ is equivalent to $\Pi(z, p_1, p_2, \Delta)$ having decreasing differences in z and Δ. Therefore, we will prove that, for $z_1 > z_2$, the difference $\Pi(z_1, p_1, p_2, \Delta) - \Pi(z_2, p_1, p_2, \Delta)$ is decreasing in Δ.

$$\begin{aligned} &\frac{\partial \left(\Pi(z_1, p_1, p_2, \Delta) - \Pi(z_2, p_1, p_2, \Delta)\right)}{\partial \Delta} = \\ - \quad &\left(q_1 \overline{G}_1(z_1) \overline{F}_1(p_1) + q_2 \overline{G}_2(z_1) \overline{F}_2(p_1)\right) - \left(q_1 G_1(z_1) \overline{F}_1(p_2) + q_2 G_2(z_1) \overline{F}_2(p_2)\right) \\ + \quad &\left(q_1 \overline{G}_1(z_2) \overline{F}_1(p_1) + q_2 \overline{G}_2(z_2) \overline{F}_2(p_1)\right) + \left(q_1 G_1(z_2) \overline{F}_1(p_2) + q_2 G_2(z_2) \overline{F}_2(p_2)\right) \end{aligned}$$

Noticing that $\overline{G}_i(z_1) - \overline{G}_i(z_2) = G_i(z_2) - G_i(z_1)$ and rearranging the terms:

$$\begin{aligned} &\frac{\partial \left(\Pi(z_1, p_1, p_2, \Delta) - \Pi(z_2, p_1, p_2, \Delta)\right)}{\partial \Delta} = \\ &q_1 \left(G_1(z_2) - G_1(z_1)\right)\left(-\overline{F}_1(p_1) + \overline{F}_1(p_2)\right) + q_2 \left(G_2(z_2) - G_2(z_1)\right)\left(-\overline{F}_2(p_1) + \overline{F}_2(p_2)\right) \end{aligned} \quad \text{(A-14)}$$

Observe that the right-hand side of the above equality is negative since $z_1 > z_2$ and $p_1 > p_2$, which concludes the proof. □

Lemma A-3 *Define* $\widetilde{\Pi}_j(z,p,\Delta) := \beta_j(z,p)(p-\Delta)$, $j=1,2$, *where* $\beta_j(z,p)$ *is as defined by* (A-4) *in Appendix C. Let* $\widetilde{p}_j(z,\Delta)$ *denote the smallest optimizer of* $\widetilde{\Pi}_j(z,p,\Delta)$. *Define* $\overline{\Pi}_i(p,\Delta) := (p-\Delta)\overline{F}_i(p)$, $i=1,2$. *Let* $\overline{p}_i(\Delta)$ *denote the optimizer of* $\overline{\Pi}_i(p)$. *Then:*
(*a*) $\overline{p}_1(\Delta) \geq \widetilde{p}_j(z,\Delta) \geq \overline{p}_2(\Delta)$.
(*b*) *At a fixed* $\Delta \geq 0$ *and* $z \geq 0$, $\widetilde{\Pi}_1(z,p,\Delta) - \widetilde{\Pi}_2(z,p,\Delta)$ *is increasing in* p *for* $p \in [\overline{p}_2(\Delta), \overline{p}_1(\Delta)]$.
(*c*) *At a fixed* $\Delta \geq 0$, $\widetilde{\Pi}_j(z,p,\Delta)$ *is supermodular in* z *and* p *for* $z \geq 0$ *and* $p \in [\overline{p}_2(\Delta), \overline{p}_1(\Delta)]$.

Proof of Lemma A-3:
Proof of (a): We can write

$$\widetilde{\Pi}_j(z,p,\Delta) = \widetilde{q}_{1j}(z)\overline{\Pi}_1(p,\Delta) + \widetilde{q}_{2j}(z)\overline{\Pi}_2(p,\Delta), j=1,2$$

We already showed in Lemma A-1(a) that $\overline{\Pi}_i(p,\Delta)$, $i=1,2$ are unimodal in p and $\overline{p}_1(\Delta) \geq \overline{p}_2(\Delta)$. Therefore, $\widetilde{\Pi}_j(z,p,\Delta)$ is increasing in p for $p \leq \overline{p}_2(\Delta)$ and decreasing in p for $p \geq \overline{p}_1(\Delta)$. Hence, the optimizer of $\widetilde{\Pi}_j(z,p,\Delta)$, denoted by $\widetilde{p}_j(z,\Delta)$, must be in $[\overline{p}_2(\Delta), \overline{p}_1(\Delta)]$.

Proof of (b): Note that

$$\frac{\partial(\widetilde{\Pi}_1(z,p,\Delta) - \widetilde{\Pi}_2(z,p,\Delta))}{\partial p} = (\beta_1(z,p) - \beta_2(z,p)) + \left(\frac{\partial\beta_1(z,p)}{\partial p} - \frac{\partial\beta_2(z,p)}{\partial p}\right)(p-\Delta)$$

where

$$\beta_j(z,p) = \widetilde{q}_{1j}(z)\overline{F}_1(p) + \widetilde{q}_{2j}(z)\overline{F}_2(p) \qquad \text{(A-15)}$$

$$\frac{\partial\beta_j(z,p)}{\partial p} = -\widetilde{q}_{1j}(z)f_1(p) - \widetilde{q}_{2j}(z)f_2(p) \qquad \text{(A-16)}$$

Therefore:

$$\begin{aligned}
\frac{\partial(\widetilde{\Pi}_1(z,p,\Delta) - \widetilde{\Pi}_2(z,p,\Delta))}{\partial p} &= (\widetilde{q}_{11}(z) - \widetilde{q}_{12}(z))\left[\overline{F}_1(p) - (p-\Delta)f_1(p)\right] \\
&+ (\widetilde{q}_{21}(z) - \widetilde{q}_{22}(z))\left[\overline{F}_2(p) - (p-\Delta)f_2(p)\right] \\
&= (\widetilde{q}_{11}(z) - \widetilde{q}_{12}(z)) \\
&\times \left\{\left[\overline{F}_1(p) - (p-\Delta)f_1(p)\right] - \left[\overline{F}_2(p) - (p-\Delta)f_2(p)\right]\right\}
\end{aligned}$$

where the last equality follows from $\widetilde{q}_{1j}(z) + \widetilde{q}_{2j}(z) = 1$. Observe that

$$\overline{F}_1(p) - (p-\Delta)f_1(p) = \frac{\partial\overline{\Pi}_1(p,\Delta)}{\partial p} \geq 0 \text{ for } p \leq \overline{p}_1(\Delta),$$

and

$$\overline{F}_2(p) - (p-\Delta)f_2(p) = \frac{\partial\overline{\Pi}_2(p,\Delta)}{\partial p} \leq 0 \text{ for } p \geq \overline{p}_2(\Delta).$$

Therefore, we will conclude the proof if we can show that $\widetilde{q}_{11} - \widetilde{q}_{12} \geq 0$. Now:

$$\widetilde{q}_{11}(z) - \widetilde{q}_{12}(z) = \frac{q_1 q_2 (G_2(z) - G_1(z))}{(q_1 \overline{G}_1(z) + q_2 \overline{G}_2(z))(q_1 G_1(z) + q_2 G_2(z))} \geq 0$$

where the inequality follows from the fact that $G_1(\cdot) < G_2(\cdot)$, because $G_1 \geq_{lr} G_2$ by assumption $(A5)$.

Proof of (c): First, note that $\widetilde{\Pi}_j(z, p, \Delta)$ being supermodular in z and p is equivalent to $\widetilde{\Pi}_j(z, p, \Delta)$ having increasing differences in z and p. Therefore, we will prove that, for $z_1 > z_2$, the difference $\widetilde{\Pi}_j(z_1, p, \Delta) - \widetilde{\Pi}_j(z_2, p, \Delta)$ is increasing in p. Note that

$$\frac{\partial \left(\widetilde{\Pi}_j(z_1, p, \Delta) - \widetilde{\Pi}_j(z_2, p, \Delta) \right)}{\partial p} = (p - \Delta) \left(\frac{\partial \beta_j(z_1, p)}{\partial p} - \frac{\partial \beta_j(z_2, p)}{\partial p} \right) + \beta_j(z_1, p) - \beta_j(z_2, p) \quad \text{(A-17)}$$

Substituting from (A-15) and (A-16) in (A-17) and rearranging the terms, we obtain:

$$\begin{aligned} \frac{\partial \left(\widetilde{\Pi}_j(z_1, p, \Delta) - \widetilde{\Pi}_j(z_2, p, \Delta) \right)}{\partial p} &= \left[-(p - \Delta) f_1(p) + \overline{F}_1(p) \right] (\widetilde{q}_{1j}(z_1, p) - \widetilde{q}_{1j}(z_2, p)) \\ &+ \left[-(p - \Delta) f_2(p) + \overline{F}_2(p) \right] (\widetilde{q}_{2j}(z_1, p) - \widetilde{q}_{2j}(z_2, p)) \end{aligned}$$

As before, observe that $\overline{F}_1(p) - (p - \Delta) f_1(p) \geq 0$ for $p \leq \overline{p}_1(\Delta)$, and $\overline{F}_2(p) - (p - \Delta) f_2(p) \leq 0$ for $p \geq \overline{p}_2(\Delta)$. Furthermore, note that $\widetilde{q}_{1j}(z_1, p) \geq \widetilde{q}_{1j}(z_2, p)$ for $j = 1, 2$ and $\widetilde{q}_{2j}(z_1, p) \leq \widetilde{q}_{2j}(z_2, p)$ for $j = 1, 2$ (by Lemma A-4). Therefore,the right-hand side of the above equality is non-negative, which concludes the proof. □

Lemma A-4 *For $z_1 > z_2$, we have $\widetilde{q}_{1j}(z_1) \geq \widetilde{q}_{1j}(z_2)$ and $\widetilde{q}_{2j}(z_1) \leq \widetilde{q}_{2j}(z_2)$, where $\widetilde{q}_{ij}(z)$ is as defined by (A-3) in Appendix C.*

Proof of Lemma A-4: Note from (A-3) that

$$\widetilde{q}_{1j}(z_1) - \widetilde{q}_{1j}(z_2) = \frac{q_1 q_2 \left(\overline{G}_1(z_1) \overline{G}_2(z_2) - \overline{G}_2(z_1) \overline{G}_1(z_2) \right)}{\left(q_1 \overline{G}_1(z_1) + q_2 \overline{G}_2(z_1) \right) \left(q_1 \overline{G}_1(z_2) + q_2 \overline{G}_2(z_2) \right)}$$

Therefore, we will conclude the proof if we can show that $\overline{G}_1(z_1) \overline{G}_2(z_2) - \overline{G}_2(z_1) \overline{G}_1(z_2) \geq 0$. Now:

$$\overline{G}_1(z_1) \overline{G}_2(z_2) - \overline{G}_2(z_1) \overline{G}_1(z_2) = \overline{G}_1(z_1)(\overline{G}_2(z_2) - \overline{G}_2(z_1)) - \overline{G}_2(z_1)(\overline{G}_1(z_2) - \overline{G}_1(z_1))$$

To conclude the proof, we note that S_1 with cdf G_1 dominates S_2 with cdf G_2 in likelihood ratio ordering. We then apply the definition of likelihood ratio ordering given by (A-2) in Appendix B, by setting $A = \{z : z_2 \le z \le z_1\}$ and $B = \{z : z \ge z_1\}$. The proof of $\widetilde{q}_{2j}(z_1) \le \widetilde{q}_{2j}(z_2)$ follows from symmetric arguments. □

Lemma A-5 *For $\Delta_t^{FP}(y)$ as defined by (6), we have:*
(a) $\Delta_t^{FP}(y) \ge \Delta_t^{FP}(y+1)$ for $t \ge 1$ and $y \ge 1$, and
(b) $\Delta_{t+1}^{FP}(y) \ge \Delta_t^{FP}(y)$ for $t \ge 1$ and $y \ge 1$.

Proof of Lemma A-5: The results follows by slight modifications to the proof of Lemma 2 in Bitran and Mondschein (1993). □

References

Aydin, G. & Ziya, S. (2008), 'Pricing promotional products under upselling', *Manufacturing and Service Operations Management* **10**, 360–376.

Bitran, G. R. & Mondschein, S. V. (1993), Pricing perishable products: An application to the retail industry. Working Paper, MIT, Sloan School of Management.

Lariviere, M. A. & Porteus, E. L. (2001), 'Selling to the newsvendor: An analysis of price-only contracts', *Manufacturing and Service Operations Management* **3**, 293–305.

Muller, A. & Stoyan, D. (2002), *Comparison Methods for Stochastic Models and Risks*, Wiley, New York.

Shaked, M. & Shanthikumar, J. G. (2007), *Stochastic Orders*, Springer, New York.