Appendix

Pricing and Capacity Allocation for Shared Services:

Technical Online Appendix

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Proof of Proposition 1 The prices (p_1, p_2) are simultaneously announced. Let us first assume that we know λ_1, λ_2. A player with valuation x_1 from class–1 will then join the system after observing the price p_1, if $U_1(x_1, \lambda_1, \lambda_2) \geq p_1$ and a player with valuation x_2 from class–2 will then join the system after observing the price p_2, if $U_2(x_2, \lambda_2, \lambda_1) \geq p_2$. Note also that if she deviates from her strategy, and does not join the system, given λ_1, λ_2, then she gains utility $0(< p_i)$. This implies, equivalently, that she will join the system if

$$\begin{aligned} x_1 + b_1 \frac{\lambda_2}{\lambda_1 + \lambda_2} + c\left(\frac{\lambda_1 + \lambda_2}{K}\right) &\geq p_1 \\ x_2 + b_2 \frac{\lambda_1}{\lambda_1 + \lambda_2} + c\left(\frac{\lambda_1 + \lambda_2}{K}\right) &\geq p_2 \end{aligned}$$

where $0 \leq x_1 \leq 1$ and $a \leq x_2 \leq 1 + a$. Define

$$\begin{aligned} x_1^* &= p_1 - b_1 \frac{\lambda_2}{\lambda_1 + \lambda_2} - c\left(\frac{\lambda_1 + \lambda_2}{K}\right), \\ x_2^* &= p_2 - b_2 \frac{\lambda_1}{\lambda_1 + \lambda_2} - c\left(\frac{\lambda_1 + \lambda_2}{K}\right) \end{aligned}$$

Then the customer will join if $x_i \geq x_i^*$, $i = 1, 2$, i.e.,

$$s_1(x_1) = \begin{cases} 1 \text{ , if } x_1^* \leq x_1 \leq 1 \\ 0 \text{ , else} \end{cases} \qquad s_2(x_2) = \begin{cases} 1 \text{ , if } x_2^* \leq x_2 \leq 1 + a \\ 0 \text{ , else} \end{cases}$$

We still need to show that there exists only one pair of (λ_1, λ_2) that leads to (x_1^*, x_2^*). Suppose there exists another pair (ν_1, ν_2), such that

$$\begin{aligned} x_1^* &= p_1 - b_1 \frac{\nu_2}{\nu_1 + \nu_2} - c\left(\frac{\nu_1 + \nu_2}{K}\right), \\ x_2^* &= p_2 - b_2 \frac{\nu_2}{\nu_1 + \nu_2} - c\left(\frac{\nu_1 + \nu_2}{K}\right). \end{aligned}$$

Then

$$\begin{aligned} -b_1 \frac{\lambda_2}{\lambda_1 + \lambda_2} - c\left(\frac{\lambda_1 + \lambda_2}{K}\right) &= -b_1 \frac{\nu_2}{\nu_1 + \nu_2} - c\left(\frac{\nu_1 + \nu_2}{K}\right) \\ -b_2 \frac{\lambda_1}{\lambda_1 + \lambda_2} - c\left(\frac{\lambda_1 + \lambda_2}{K}\right) &= -b_2 \frac{\nu_1}{\nu_1 + \nu_2} - c\left(\frac{\nu_1 + \nu_2}{K}\right) \end{aligned}$$

which implies that

$$\begin{aligned} -b_1 \frac{\lambda_2}{\lambda_1 + \lambda_2} - c\left(\frac{\lambda_1 + \lambda_2}{K}\right) &= -b_1 \frac{\nu_2}{\nu_1 + \nu_2} - c\left(\frac{\nu_1 + \nu_2}{K}\right) \\ \frac{-b_1\lambda_2 + b_2\lambda_1}{\lambda_1 + \lambda_2} &= \frac{-b_1\nu_2 + b_2\nu_1}{\nu_1 + \nu_2} \end{aligned}$$

and combined with $c''(\cdot) < 0$, leads to

$$\frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\nu_2}{\nu_1 + \nu_2} \text{ and } \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{\nu_1}{\nu_1 + \nu_2}$$

and so $\lambda_i = \nu_i,\ i = 1, 2$. Using also the fact that the service values follow a uniform distribution and $\lambda_1 = \Lambda\int_{x_1^*}^{1} s_1(x_1)dx_1,\ \lambda_2 = \Lambda\int_{x_2^*}^{a+1} s_2(x_1)dx_2$, we have that

$$\lambda_1^* = \Lambda(1 - x_1^*) \text{ and } \lambda_2^* = \Lambda(a + 1 - x_2^*).$$

From equation (1), there exists a marginal customer from each class whose utility will satisfy

$$x_1^* + b_1\frac{a+1-x_2^*}{a+2-x_1^*-x_2^*} + c\left(\frac{\Lambda(a+2-x_1^*-x_2^*)}{K}\right) = p_1,$$
$$x_2^* + b_2\frac{1-x_1^*}{a+2-x_1^*-x_2^*} + c\left(\frac{\Lambda(a+2-x_1^*-x_2^*)}{K}\right) = p_2.$$

The solution of this system, (x_1^*, x_2^*) will denote the Nash equilibrium (NE) of the game. Since there is a unique mapping between (x_1^*, x_2^*) and (λ_1, λ_2), the NE can be equivalently expressed in terms of (λ_1, λ_2) and the equilibrium prices will be derived as follows

$$p_1(\lambda_1, \lambda_2) = 1 - \lambda_1/\Lambda + b_1\lambda_2/(\lambda_1+\lambda_2) + c\left((\lambda_1+\lambda_2)/K\right), \quad (8)$$
$$p_2(\lambda_2, \lambda_1) = 1 + a - \lambda_2/\Lambda + b_2\lambda_1/(\lambda_1+\lambda_2) + c\left((\lambda_1+\lambda_2)/K\right). \quad (9)$$

□

Proof of Lemma 1 We first show part (i). Note that solution $(0, 0)$ is never optimal. Consider solution $(0, \epsilon)$ with $\epsilon > 0$ and sufficiently small. This solution yields revenue $R(0, \epsilon) = \epsilon(1 + c(\epsilon/K)) - \epsilon^2/\Lambda$. Now $\lim_{\epsilon\downarrow 0} R(0, \epsilon) > \lim_{\epsilon\downarrow 0} \epsilon - \epsilon^2/\Lambda = 0$ (recall that $c(0) > -1$). Therefore, $\lambda_1^* = \lambda_2^* \Rightarrow \lambda_1^*\lambda_2^* > 0$.

To show that $\lambda_1^*\lambda_2^* > 0 \Rightarrow \lambda_1^* = \lambda_2^*$, consider problem (P1) and note that $R(\lambda_1, \lambda_2) = \lambda_1\{1 - \lambda_1/\Lambda + b_1\lambda_2/(\lambda_1+\lambda_2) + c[(\lambda_1+\lambda_2)/K]\} + \lambda_2\{1 + a - \lambda_2/\Lambda + b_2\lambda_1/(\lambda_1+\lambda_2) + c[(\lambda_1+\lambda_2)/K]\}$. Let $\{\mu_i \geq 0 : i = 1, 2, 3\}$ be KKT multipliers for constraints $-(\lambda_1+\lambda_2) + K \geq 0,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0$, respectively. A candidate optimal solution must satisfy $\mu_1(-\lambda_1 - \lambda_2 + K) = 0,\ \mu_2\lambda_1 = 0,\ \mu_3\lambda_2 = 0$, and the following stationarity conditions,

$$1 - 2\lambda_1/\Lambda + b\lambda_2^2/(\lambda_1+\lambda_2)^2 - \mu_1 + \mu_2 + c[(\lambda_1+\lambda_2)/K] + (\lambda_1+\lambda_2)c'[(\lambda_1+\lambda_2)/K]/K = 0, \quad (10)$$

$$1 + a - 2\lambda_2/\Lambda + b\lambda_1^2/(\lambda_1+\lambda_2)^2 - \mu_1 + \mu_3 + c[(\lambda_1+\lambda_2)/K] + (\lambda_1+\lambda_2)c'[(\lambda_1+\lambda_2)/K]/K = 0. \quad (11)$$

Let $a = 0,\ \mu_1 = \mu_2 = \mu_3 = 0$, and subtract equation (10) from equation (11), which yields

$$(\lambda_1 - \lambda_2)[2/\Lambda + b/(\lambda_1+\lambda_2)] = 0.$$

Therefore, either $\lambda_1 = \lambda_2$ or $\lambda_1 + \lambda_2 = -b\Lambda/2$. Suppose $\lambda_1 + \lambda_2 = -b\Lambda/2$, which implies $b < 0$. Likewise, letting $a = 0,\ \mu_1 = \mu_2 = \mu_3 = 0$, and adding up equations (10) and (11) yields $\lambda_1\lambda_2 = (1 + b + c(-b\Lambda/(2K)) - b\Lambda/(2K)c'(-b\Lambda/(2K)))b\Lambda^2/4$, which in turn implies $1 + b + c(-b\Lambda/(2K)) - b\Lambda/(2K)c'(-b\Lambda/(2K)) < 0$. Substituting $\lambda_1 + \lambda_2 = -b\Lambda/2$ into (10) leads to $1 + b + c(-b\Lambda/(2K)) - b\Lambda/(2K)c'(-b\Lambda/(2K)) = -2\lambda_2(1 + 2\lambda_2/(b\Lambda))/\Lambda$ that should be negative. This implies that $1 + 2\lambda_2/(b\Lambda) > 0$. Similarly, if we substitute $\lambda_1 + \lambda_2 = -b\Lambda/2$ into (11), we have that $1 + 2\lambda_1/(b\Lambda) > 0$. Adding these two implies that $1 + 2(\lambda_1+\lambda_2)/(b\Lambda) > 0$ that leads to a contradiction since we assumed that $\lambda_1 + \lambda_2 = -b\Lambda/2$. Therefore, $\lambda_1^*\lambda_2^* > 0 \Rightarrow \lambda_1^* = \lambda_2^*$.

Part (ii) follows directly from the fact that $R(\lambda,0)=R(0,\lambda)$ if $a=0$.

We now show part (iii). First, note that that $\lambda_1=\lambda_2$ cannot be optimal if $a>0$, because the system of equations (10)–(11) does not have a solution in that case. Now consider solution (x,y), $x>y$. This solution yields revenue $R(x,y)=x(1-x/\Lambda)+y(1+a-y/\Lambda)+bxy/(x+y)+(x+y)c((x+y)/K)$. Likewise, $R(y,x)=y(1-y/\Lambda)+x(1+a-x/\Lambda)+bxy/(x+y)+(x+y)c((x+y)/K)$. Note that $R(y,x)-R(x,y)=a(x-y)>0$. Thus, we must have $\lambda_2^*>\lambda_1^*$ if $a>0$. □

Proof of Proposition 2 We use the same notation as in the proof of Lemma 1. First, recall that solution $(0,0)$ is never optimal. Second, we know from Lemma 1 that $\lambda_2^*\geq\lambda_1^*\geq 0$, therefore, $\lambda_2^*>0$, and thus, $\mu_3^*=0$ always. Hereafter, we remove the complementary slackness constraint $\mu_3\lambda_2=0$ from further consideration.

To show part (i), consider a feasible solution to (P1) where $\lambda_1>0$. Note that for sufficiently negative values of b such a solution cannot be optimal because $\lim_{b\to-\infty}R(\lambda_1,\lambda_2)=-\infty$ if $\lambda_1>0$. Therefore, for sufficiently negative values of b, $\lambda_1^*=0$. Similarly, for sufficiently positive values of b we must have $\lambda_1^*>0$ because $\lim_{b\to\infty}R(\lambda_1,\lambda_2)=\infty$ if $\lambda_1>0$. Suppose now that $\lambda_1^*>0,\lambda_2^*=\lambda^{'}$ at $b=b^{'}$, but $\lambda_1^*=0,\lambda_2^*=\lambda^{''}$ at $b=b^{''}>b^{'}$. However, by the Envelope Theorem, $\partial R(\lambda_1^*,\lambda')/\partial b=\lambda_1^*\lambda'/(\lambda_1^*+\lambda')>0$, whereas $\partial R(0,\lambda^{''})/\partial b=0$. Therefore, solution $(\lambda_1^*,\lambda^{'})$ could not have been optimal at $b=b'$, and we have arrived at a contradiction. We thus conclude that there exists threshold b^*, which depends on K in general, such that $\lambda_1^*=0$ if $b\in(-\infty,b^*(K)]$, and $\lambda_1^*>0$ if $b\in(b^*(K),\infty)$.

For the proof of part (ii), it suffices to show that function $R(\lambda_1^*,\lambda_2^*)\uparrow K$ if $\lambda_1^*+\lambda_2^*=K$ and $K\leq\min(\Lambda[1+a+c(1)]/2,2[1+c(1)]\Lambda/3)$. By the Envelope Theorem, $\partial R(\lambda_1^*,\lambda_2^*)/\partial K=\mu_1^*-(\lambda_1^*+\lambda_2^*)^2c'[(\lambda_1^*+\lambda_2^*)/K]/K^2$, where μ_1^* can be calculated using equation (11). If $\lambda_1^*=0$, then $\mu_1^*=1+a+c(1)+c'(1)-2K/\Lambda$; thus, $\partial R(\lambda_1^*,\lambda_2^*)/\partial K=1+a+c(1)-2K/\Lambda$; therefore, $R(\lambda_1^*,\lambda_2^*)\uparrow K$ if $K\leq\Lambda[1+a+c(1)]/2$. If $\lambda_1^*>0$, then $\partial R(\lambda_1^*,\lambda_2^*)/\partial K=1+a+c(1)-2\lambda_2^*/\Lambda+b(K-\lambda_2^*)^2/K^2$. In addition, equations (10) and (11) jointly yield the solution $\lambda_1^*=K/2-aK\Lambda/[2(2K+b\Lambda)]$, $\lambda_2^*=K/2+aK\Lambda/[2(2K+b\Lambda)]$; therefore, we require that $b>a-2K/\Lambda\geq-2K/\Lambda$ so that $\lambda_1^*>0$. (If $b<-a-2K/\Lambda$, the solution in question is a local minimum.) Because $\lambda_2^*\geq K/2$ and $\partial R(\lambda_1^*,\lambda_2^*)/\partial K$ is a second–order polynomial wrt λ_2^*, for our purposes it suffices to show that $\lim_{\lambda_2^*\to K^-}\partial R(\lambda_1^*,\lambda_2^*)/\partial K\geq 0$ and that $\partial R(\lambda_1^*,\lambda_2^*)/\partial K|_{a=0,\lambda_1^*=K/2}\geq 0$. To that end, first note that $\lim_{\lambda_2^*\to K^-}\partial R(\lambda_1^*,\lambda_2^*)/\partial K=1+a+c(1)-2K/\Lambda\geq 0$, where the last inequality is because $K\leq\Lambda[1+a+c(1)]/2$. Furthermore, $\partial R(\lambda_1^*,\lambda_2^*)/\partial K|_{a=0,\lambda_1^*=K/2}=b/4-K/\Lambda+1+c(1)>-3K/(2\Lambda)+1+c(1)\geq 0$, where the last two inequalities are because $b>-2K/\Lambda$ and $K\leq 2[(1+c(1)]\Lambda/3$, respectively.

Part (iii) follows immediately from the fact that $\lambda_1^*+\lambda_2^*\leq 2\Lambda$.

To show part (iv), first note that parts (ii) and (iii) establish that there exists at least one switching point at which the system goes from being full to being not full. We show next that if the conditions of part (iv) hold, the switching point is unique. Letting $\lambda_1+\lambda_2=K$ in equation (11) yields $\mu_1^*=1+a+c(1)+c'(1)-2\lambda_2^*/\Lambda+b(\lambda_1^*)^2/K^2$. To show that the switching point is unique, it suffices to show that $\mu_1^*(K)\downarrow K$. To this end, first suppose that $b<<0\Rightarrow\lambda_1^*=0\ \forall K\geq 0$. In this case, $\mu_1^*(K)=1+a+c(1)+c'(1)-2K/\Lambda\Rightarrow\partial\mu_1^*(K)/\partial K=-2K/\Lambda<0$. Next, suppose $b>>0\Rightarrow\lambda_1^*>0\ \forall K\geq 0$. Letting $\lambda_1+\lambda_2=K$, $\mu_2=0$ in the system of equations (10)–(11) yields the solution $\lambda_1^*=K/2-aK\Lambda/[2(2K+b\Lambda)],\lambda_2^*=K/2+aK\Lambda/[2(2K+b\Lambda)]$. Straightforward calculus yields $\partial\mu_1^*(K)/\partial K=-1/\Lambda-ba^2\Lambda^2/(2K+b\Lambda)^3<0$. □

Proof of Lemma 3 The proof of part (ii) is straightforward. We show here the proof for part (i); in particular, we will show that $\lambda_1^* \geq \lambda_2^*$ if $b_1 \geq b_2$ and $\lambda_2^* \geq \lambda_1^*$ if $b_2 \geq b_1$ and $\lambda_1^*\lambda_2^* > 0$. To that end, consider problem (P2′) and note that $R(\lambda_1, \lambda_2) = \lambda_1\{1 - \lambda_1/\Lambda + b_1\lambda_2/(\lambda_1+\lambda_2) + c[(\lambda_1+\lambda_2)/K]\} + \lambda_2\{1 - \lambda_2/\Lambda + b_2\lambda_1/(\lambda_1+\lambda_2) + c[(\lambda_1+\lambda_2)/K] + a\}$. Let $\{\mu_i \geq 0 : i = 1,2,3\}$ be KKT multipliers for constraints $-(\lambda_1 + \lambda_2) + K \geq 0$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, respectively. Let $g \in \mathbb{R}$ be the Lagrange multiplier for constraint $[b_1/(\lambda_1+\lambda_2) + 1/\Lambda]\lambda_2 - [b_2/(\lambda_1+\lambda_2) + 1/\Lambda]\lambda_1 - a = 0$. A candidate optimal solution must satisfy $\mu_1(-\lambda_1 - \lambda_2 + K) = 0$, $\mu_2\lambda_1 = 0$, $\mu_3\lambda_2 = 0$, $[b_1/(\lambda_1+\lambda_2) + 1/\Lambda]\lambda_2 - [b_2/(\lambda_1+\lambda_2) + 1/\Lambda]\lambda_1 - a = 0$, and the following stationarity conditions.

$$1 - 2\lambda_1/\Lambda + b\lambda_2^2/(\lambda_1+\lambda_2)^2 - \mu_1 + \mu_2 + c[(\lambda_1+\lambda_2)/K] + (\lambda_1+\lambda_2)c'[(\lambda_1+\lambda_2)/K]/K$$
$$-g[b\lambda_2/(\lambda_1+\lambda_2)^2 + 1/\Lambda] = 0, \tag{12}$$
$$1 + a - 2\lambda_2/\Lambda + b\lambda_1^2/(\lambda_1+\lambda_2)^2 - \mu_1 + \mu_3 + c[(\lambda_1+\lambda_2)/K] + (\lambda_1+\lambda_2)c'[(\lambda_1+\lambda_2)/K]/K$$
$$+g[b\lambda_1/(\lambda_1+\lambda_2)^2 + 1/\Lambda] = 0. \tag{13}$$

Let $a = 0$, $\mu_1 = \mu_2 = \mu_3 = 0$, and subtract equation (12) from equation (13), which yields

$$(\lambda_1 - \lambda_2 + g)[2/\Lambda + b/(\lambda_1+\lambda_2)] = 0.$$

Therefore, either $g = \lambda_2 - \lambda_1$ or $\lambda_1 + \lambda_2 = -b\Lambda/2$. If $b_1 = b_2$, we know from part (i) of Lemma 1 that $\lambda_1^* = \lambda_2^*$, which remains an optimal solution here because it satisfies the single–price constraint (4) if $a = 0$. Thus, we assume—without loss of generality—that $b_1 > b_2$ for the remainder of the proof. Suppose that $\lambda_1 + \lambda_2 = -b\Lambda/2$. Then, the last equation together with equation (4) for $a = 0$ yield $\lambda_1 + \lambda_2 = 0$, which is not possible unless $\lambda_1 = \lambda_2 = 0$. However, we know from part (i) of Lemma 1 that solution $(0,0)$ is never optimal. Therefore, $g^* = \lambda_2^* - \lambda_1^*$ when $\lambda_1^*\lambda_2^* > 0$ and $a = 0$.

Ignoring the non-binding constraints, the Lagrange function for problem (P2′) and $a = 0$ is $\mathcal{L}(\lambda_1, \lambda_2) = \lambda_1\{1 - \lambda_1/\Lambda + b_1\lambda_2/(\lambda_1+\lambda_2) + c[(\lambda_1+\lambda_2)/K]\} + \lambda_2\{1 - \lambda_2/\Lambda + b_2\lambda_1/(\lambda_1+\lambda_2) + c[(\lambda_1+\lambda_2)/K]\} + g\{[b_1/(\lambda_1+\lambda_2) + 1/\Lambda]\lambda_2 - [b_2/(\lambda_1+\lambda_2) + 1/\Lambda]\lambda_1\}$. Let $\mathcal{L}^*(\lambda_1^*, \lambda_2^*) \equiv \max \mathcal{L}(\lambda_1, \lambda_2)$. By the Envelope Theorem, $\partial\mathcal{L}^*(\lambda_1^*, \lambda_2^*)/\partial b_2 = \lambda_1^*\lambda_2^*/(\lambda_1^* + \lambda_2^*) - g^*\lambda_1^*/(\lambda_2^* + \lambda_2^*) = (\lambda_1^*)^2/(\lambda_1^* + \lambda_2^*)$, where the second equality is because $g^* = \lambda_2^* - \lambda_1^*$. Likewise, $\partial\mathcal{L}^*(\lambda_1^*, \lambda_2^*)/\partial b_1 = (\lambda_2^*)^2/(\lambda_1^* + \lambda_2^*)$.

To complete the proof, suppose wlog that $b_1 > b_2$ and that solution $(\lambda_1'(b_1), \lambda_2'(b_1))$, where $\lambda_1' < \lambda_2'$, satisfies constraint (4) and the stationarity conditions (12)-(13). By applying the expressions that the Envelope Theorem stipulates, $R(\lambda_1'(b_1), \lambda_2'(b_1)) \uparrow b_1$; therefore, solution $(\lambda_1'(b_1), \lambda_2'(b_1))$, where $0 < \lambda_1'(b_1) < \lambda_2'(b_1)$, will be optimal if b_1 is sufficiently large. Note, however, that if $b_1 > b_2 \geq 0$ or $b_1 \geq 0 > b_2$ and $\lambda_2 > \lambda_1$, constraint (4) cannot be satisfied. Therefore, we have reached a contradiction; a solution such that $\lambda_1 < \lambda_2$ cannot be optimal if $b_1 \geq b_2$. □

Proof of Proposition 3 We use the same notation as in the proof of Lemma 3. First, note that solution $(0,0)$ is never optimal. Second, we know from part (iii) of Lemma 1 that in an exclusive system, $\lambda_2^* > \lambda_1^* = 0$ if $a > 0$. If $a = 0$, $\lambda_2^* > \lambda_1^* = 0$ is still optimal. Therefore, $\lambda_2^* > 0$ and thus, $\mu_3^* = 0$ always. Hereafter, we remove the complementary slackness constraint $\mu_3\lambda_2 = 0$ from further consideration.

To prove the first statement of part (i), consider a feasible solution to (P2′) such that $\lambda_1 > 0$. Note that for sufficiently negative values of b such a solution cannot be optimal because $\lim_{b\to-\infty} R(\lambda_1,\lambda_2) = -\infty$ if $\lambda_1 > 0$. Therefore, for sufficiently negative values of b, $\lambda_1^* = 0$. Similarly, for sufficiently positive values of b we must have $\lambda_1^* > 0$ because $\lim_{b\to\infty} R(\lambda_1,\lambda_2) = \infty$ if $\lambda_1 > 0$.

Ignoring the non-binding constraints, the Lagrange function for problem (P2′) is $\mathcal{L}(\lambda_1,\lambda_2) = \lambda_1\{1-\lambda_1/\Lambda + b_1\lambda_2/(\lambda_1+\lambda_2) + c[(\lambda_1+\lambda_2)/K]\} + \lambda_2\{1+a-\lambda_2/\Lambda + b_2\lambda_1/(\lambda_1+\lambda_2) + c[(\lambda_1+\lambda_2)/K]\} + \mu_1(-\lambda_1-\lambda_2+K) + g\{[b_1/(\lambda_1+\lambda_2)+1/\Lambda]\lambda_2 - [b_2/(\lambda_1+\lambda_2)+1/\Lambda]\lambda_1 - a\}$. Let $\mathcal{L}^*(\lambda_1^*,\lambda_2^*) \equiv \max \mathcal{L}(\lambda_1,\lambda_2)$. By the Envelope Theorem, $\partial\mathcal{L}^*(\lambda_1^*,\lambda_2^*)/\partial b_1 = (\lambda_1^*\lambda_2^* + g^*\lambda_2^*)/(\lambda_1^*+\lambda_2^*) = \lambda_2^*(\lambda_1^*+g^*)/(\lambda_1^*+\lambda_2^*)$. Similarly, $\partial\mathcal{L}^*(\lambda_1^*,\lambda_2^*)/\partial b_2 = \lambda_1^*(\lambda_2^*-g^*)/(\lambda_1^*+\lambda_2^*)$. First, we will show that $\lambda_1^*+g^* \geq 0$. Suppose, to the contrary, that $\lambda_1^*+g^* < 0$ at the optimal solution. This necessarily implies that for this particular solution $R(\lambda_1^*,\lambda_2^*) \downarrow b_1$, $b_1 \in (-\infty,\infty)$. Now recall that if $\lambda_1 > 0$, $\lim_{b_1\to-\infty} R(\lambda_1,\lambda_2) = -\infty$. Therefore, a solution such that $\lambda_1^*+g^* < 0$ cannot be optimal at any value of b_1. By the same logic, it must be true that $\lambda_2^* - g^* \geq 0$ at the optimal solution. Therefore, $R(\lambda_1^*,\lambda_2^*) \uparrow b_1$ and $R(\lambda_1^*,\lambda_2^*) \uparrow b_2$. On the other hand, the revenue of an exclusive system is not a function of b_1, b_2; therefore, we conclude that there exists threshold b^*, which depends on K in general, such that $\lambda_1^* = 0$ if $b \in (-\infty, b^*(K)]$, and $\lambda_1^* > 0$ if $b \in (b^*(K),\infty)$. This completes the proof of the first statement of part (i).

To prove the second statement of part (i), let $a = 0$, $b_1 + b_2 = \tilde{b}$, i.e., b is constant, and assume wlog that $b_1 \geq b_2$, in which case $\lambda_1^* \geq \lambda_2^*$ by Lemma 3. Also, recall from the proof of Lemma 3 that $g^* = \lambda_2^* - \lambda_1^*$ when $\lambda_2^*\lambda_1^* > 0$ and $a = 0$. Then, $\partial\mathcal{L}^*(\lambda_1^*,\lambda_2^*)/\partial b_1 - \partial\mathcal{L}^*(\lambda_1^*,\lambda_2^*)/\partial b_2 = [(\lambda_2^*)^2 - (\lambda_1^*)^2]/(\lambda_1^*+\lambda_2^*) \leq 0$. Therefore, $\mathcal{L}^*(\lambda_1^*,\lambda_2^*) \downarrow \Delta b$. Because the revenue of an exclusive system is not a function of b_1, b_2, there must exist threshold Δb^*, which depends on K in general, such that $\lambda_1^* > 0$ if $b \in (0, \Delta b^*(K)]$, and $\lambda_1^* = 0$ if $\Delta b \in (\Delta b^*(K),\infty)$.

For the proof of part (ii), it suffices to show that function $R(\lambda_1^*,\lambda_2^*) \uparrow K$ if $\lambda_1^*+\lambda_2^* = K$ and $K \leq \min(\Lambda[1+c(1)], \Lambda[1+a+c(1)]/2)$. By the Envelope Theorem, $\partial R(\lambda_1^*,\lambda_2^*)/\partial K = \mu_1^* - (\lambda_1^*+\lambda_2^*)^2 c'[(\lambda_1^*+\lambda_2^*)/K]/K^2$, where μ_1^* can be calculated using equation (13). If $\lambda_1^* = 0$, then $\mu_1^* = 1 + a + c(1) + c'(1) - 2K/\Lambda$; thus, $\partial R(\lambda_1^*,\lambda_2^*)/\partial K = 1 + a + c(1) - 2K/\Lambda$; therefore, $R(\lambda_1^*,\lambda_2^*) \uparrow K$ if $K \leq \Lambda[1+a+c(1)]/2$. Consider next the case $\lambda_1^* > 0$. The fact that $\lambda_1^*+\lambda_2^* = K$ and equation (4) jointly imply the solution $\lambda_1^* = K[K+(b_1-a)\Lambda]/(2K+b\Lambda)$, $\lambda_2^* = K[K+(b_2+a)\Lambda]/(2K+b\Lambda)$; therefore, we require that $b_1 > a - K/\Lambda \geq -K/\Lambda$, $b_2 > -K/\Lambda$, $b > a - 2K/\Lambda \geq -2K/\Lambda$ so that $\lambda_1^* > 0$. Using the expressions for λ_1^*, λ_2^*, and subtracting equation (12) from (13) yields $g^* = (a+b_2-b_1)K\Lambda/(2K+b\Lambda)$. Also, by equation (13), $\mu_1^* = 1 + a + c(1) + c'(1) - 2\lambda_2^*/\Lambda + b(K-\lambda_2^*)^2/K^2 + g^*[b(K-\lambda_2^*)/K^2 + 1/\Lambda]$; thus, $\partial R(\lambda_1^*,\lambda_2^*)/\partial K = 1 + a + c(1) - 2\lambda_2^*/\Lambda + b(K-\lambda_2^*)^2/K^2 + g^*[b(K-\lambda_2^*)/K^2 + 1/\Lambda]$. Because $K > \lambda_2^* > 0$ and $\partial R(\lambda_1^*,\lambda_2^*)/\partial K$ is a second–order polynomial wrt λ_2^*, for our purposes it suffices to show that $\lim_{\lambda_2^*\to K^-} \partial R(\lambda_1^*,\lambda_2^*)/\partial K \geq 0$ and that $\lim_{\lambda_2^*\to 0^+} \partial R(\lambda_1^*,\lambda_2^*)/\partial K \geq 0$. Because $\lim_{\lambda_2^*\to K^-} g^* = (b_2+K/\Lambda)K\Lambda/(2K+b\Lambda)$, $\lim_{\lambda_2^*\to K^-} \partial R(\lambda_1^*,\lambda_2^*)/\partial K = 1 + a + c(1) - 2K/\Lambda + (b_2+K/\Lambda)K/(2K+b\Lambda) \geq 0$, where the last inequality is because $K \leq \Lambda[1+a+c(1)]/2$ and $b_2 > -K/\Lambda$. Similarly, $\lim_{\lambda_2^*\to 0^+} g^* = -K$, because $b_1 > -K/\Lambda$ and $\lambda_2^* \to 0^+$ iff $b_2 \to -K/\Lambda$ and $a \to 0$. Thus, $\lim_{\lambda_2^*\to 0^+} \partial R(\lambda_1^*,\lambda_2^*)/\partial K = 1 + c(1) - K/\Lambda \geq 0$, because $K \leq \Lambda[1+c(1)]$.

Part (iii) follows immediately from the fact that $\lambda_1^*+\lambda_2^* \leq 2\Lambda$.

To show part (iv), first note that parts (ii) and (iii) establish that there exists at least one switching point at which the system goes from being full to being not full. We show next that if the conditions of part (iv) hold, the switching point is unique. The proof for the case $b << 0 \Rightarrow \lambda_1^* = 0\ \forall K \geq 0$ can be found in the proof of part (iv) of Proposition 2. Next, suppose that $b >> 0 \Rightarrow \lambda_1^* > 0\ \forall K \geq 0$. The fact that $\lambda_1^* + \lambda_2^* = K$ and equation (4) jointly imply the solution $\lambda_1^* = K[K + (b_1 - a)\Lambda]/(2K + b\Lambda)$, $\lambda_2^* = K[K + (b_2 + a)\Lambda]/(2K + b\Lambda)$. Using these expressions for λ_1^*, λ_2^*, and subtracting equation (12) from (13) yields $g^* = (a + b_2 - b_1)K\Lambda/(2K + b\Lambda)$. Also, by equation (12), $\mu_1^* = 1 + c(1) + c'(1) - 2\lambda_1^*/\Lambda + b(K - \lambda_1^*)^2/K^2 - g^*[b(K - \lambda_1^*)/K^2 + 1/\Lambda]$. To show that the switching point is unique, it suffices to show that $\mu_1^*(K) \downarrow K$. To that end, note that

$$\frac{\partial \mu_1^*(K)}{\partial K} = \frac{-8K^3 - 6bK\Lambda(2K + b\Lambda) - 2b\Lambda^3[a(b_1 - b_2) + 2b_1b_2]}{\Lambda(2K + b\Lambda)^3}.$$

In the last fraction, the denominator is positive because $b > 0$; thus, we focus on the numerator, whose sign is ambiguous in general. First, notice that the numerator is strictly decreasing in K, because $b > 0$. Second, notice that if $a(b_1 - b_2) + 2b_1b_2 \geq 0$, then $\partial \mu_1^*(K)/\partial K < 0$, which implies that the switching point is unique. Suppose next that $a(b_1 - b_2) + 2b_1b_2 < 0$. Then, for sufficiently small values of K, $\partial \mu_1^*(K)/\partial K > 0$; therefore, as the numerator is strictly decreasing in K, there exists a unique value of K above which $\partial \mu_1^*(K)/\partial K < 0$. In addition, because $\lambda_1^* + \lambda_2^* = K$ if $K \leq \Lambda/2$, solution $(0, \Lambda(1 + a)/2)$ can only become optimal at some sufficiently large value of capacity at which $\partial \mu_1^*(K)/\partial K < 0$. As a result, the switching point is, again, unique. □

Proof of Lemma 4 For part (i), we provide the proof for the case $\lambda_1^*\lambda_2^* > 0$, $\lambda_1^* < (1 - x^*)K$, $\lambda_2^* < x^*K$. The proofs for the other cases are identical in spirit and thus omitted. Solution $\{(\lambda_1^*, \lambda_2) : \lambda_1^*\lambda_2^* > 0, \lambda_1^* < (1 - x^*)K, \lambda_2^* < x^*K\}$ satisfies the following stationarity conditions:

$$\partial R(\lambda_1, \lambda_2, x)/\partial \lambda_1 = 0 \Leftrightarrow 1 - 2\lambda_1/\Lambda + c\{\lambda_1/[(1 - x)K]\} + \lambda_1 c'\{\lambda_1/[(1 - x)K]\}/[(1 - x)K] = 0, \tag{14}$$

$$\partial R(\lambda_1, \lambda_2, x)/\partial \lambda_2 = 0 \Leftrightarrow 1 + a - 2\lambda_2/\Lambda + c[\lambda_2/(xK)] + \lambda_2 c'[\lambda_2/(xK)]/(xK) = 0, \tag{15}$$

$$\partial R(\lambda_1, \lambda_2, x)/\partial x = 0 \Leftrightarrow \lambda_1^2 c'\{\lambda_1/[(1 - x)K]\}/[(1 - x)^2K] - \lambda_2^2 c'[\lambda_2/(xK)]/(x^2K) = 0. \tag{16}$$

We first show that for any allocation $x \in (0, 1)$, there exists at most one solution $(\lambda_1^*(x), \lambda_2^*(x))$ satisfying (14) and (15). (If no such solution exists, the optimal solution must be an extreme point.) In particular, we will show that there exists at most one $\lambda_2^*(x)$ satisfying (15)—the proof that there is at most one $\lambda_1^*(x)$ satisfying (14) is very similar as the two equations differ only by constant $a \geq 0$ once $1 - x$ is replaced by x in (14).

If we let $u \equiv \lambda_2/(xK)$, the LHS of equation (15) becomes $G(u) \equiv 1 + a - 2uxK/\Lambda + c(u) + uc'(u)$. It suffices to show that $G(u) = 0$ cannot have two roots in $(0, 1)$. Note that $G(0) = 1 + a + c(0) > 0$ because $c(0) > -1$, and that $\partial G(u)/\partial u = -2xK/\Lambda + 2c'(u) + uc''(u)$. Because $-2xK/\Lambda < 0$, $c''(u) < 0 \Rightarrow uc''(u) < 0$, either $G(u) \downarrow u$ in $(0, 1)$ or $G(u) \uparrow u$ in $(0, 1)$, or $G(u) \uparrow u$ in $(0, u^*)$ and $G(u) \downarrow u$ in $(u^*, 1)$, $u^* \in (0, 1)$. Thus, $G(u)$ cannot have two roots in $(0, 1)$. Therefore, for any allocation $x \in (0, 1)$, there exists at most one optimal solution to (P3) such that $\lambda_1^*(x)\lambda_2^*(x) > 0$. To complete the proof for part (i), we note that the allocation $x = \lambda_2/(\lambda_1 + \lambda_2)$ satisfies (16) and invoke parts (ii) and (iii) of the lemma, which we show next.

For part (ii), we will show that if $\lambda_1^*\lambda_2^* > 0$, the uniquely optimal allocation satisfies $\lambda_1^*/[(1 - x^*)K] = \lambda_2^*/(x^*K) \Leftrightarrow x^* = \lambda_2^*/(\lambda_1^* + \lambda_2^*)$, i.e., the two capacity segments have the same crowding level at the optimal solution. To that end, let $\lambda_1/[(1-x)K] \equiv u_1$ and $\lambda_2/(xK) \equiv u_2$. The objective function in (P3) in terms of u_1, u_2 is

$$R(u_1,u_2) = \lambda_1 + \lambda_2 - (\lambda_1^2+\lambda_2^2)/\Lambda + \lambda_1 c(u_1) + \lambda_2 c(u_2) + \lambda_2 a.$$

In the revenue function above, fix λ_1, λ_2, where $\lambda_1\lambda_2 > 0$, and notice that only the term $\lambda_1 c(u_1) + \lambda_2 c(u_2)$ involves allocations u_1, u_2. Suppose $u_1 \neq u_2$. Because $c'' < 0$,

$$[\lambda_1/(\lambda_1+\lambda_2)]c(u_1) + [\lambda_2/(\lambda_1+\lambda_2)]c(u_2) < c[u_1\lambda_1/(\lambda_1+\lambda_2) + u_2\lambda_2/(\lambda_1+\lambda_2)]$$
$$\Leftrightarrow \lambda_1 c(u_1) + \lambda_2 c(u_2) \; < \; \lambda_1 c[u_1\lambda_1/(\lambda_1+\lambda_2) + u_2\lambda_2/(\lambda_1+\lambda_2)] + \lambda_2 c[u_1\lambda_1/(\lambda_1+\lambda_2) + u_2\lambda_2/(\lambda_1+\lambda_2)].$$

Therefore, crowding levels u_1', u_2' such that $u_1' = u_2' = u_1\lambda_1/(\lambda_1+\lambda_2) + u_2\lambda_2/(\lambda_1+\lambda_2)$ yield strictly higher revenue than crowding levels u_1, u_2. As a result, $u_1 = u_2$ at optimality.

Part (iii) follows directly from part 2 if $\lambda_1^*\lambda_2^* > 0$. If $\lambda_1^* = 0$, notice that an allocation $x^* < 1$ cannot satisfy (16), thus it cannot be optimal.

For part (iv), the optimal allocation if $a = 0$ follows directly from the fact that $x^* = \lambda_2^*/(\lambda_1^* + \lambda_2^*)$ and equations (14) and (15). Next we show that $x^*(a) \uparrow a$. Let $u^* \equiv \lambda_1^*/[(1-x^*)^*K] = \lambda_2^*/(x^*K)$ so that (14) becomes $F(u^*(x^*), x^*) \equiv 1 - 2u^*(1-x^*)K/\Lambda + c(u^*) + u^*c'(u^*) = 0$. By the Implicit Function Theorem,

$$\partial u^*(x^*)/\partial x^* = -\frac{\partial F(u^*,x^*)/\partial x^*}{\partial F(u^*,x^*)/\partial u^*} = -\frac{2u^*K/\Lambda}{2(x^*-1)K/\Lambda + 2c'(u^*) + u^*c''(u^*)}.$$

In the last fraction, note that $u^* > 0$ and $x^* < 1$. We will show that $2c'(u^*) + u^*c''(u^*) \leq 0$ so that $\partial u^*(x^*)/\partial x^* > 0$. Using straightforward calculus, $\partial^2 R(\lambda_1,\lambda_2,x)/\partial x^2|_{\lambda_1=\lambda_1^*,\lambda_2=\lambda_2^*,x=x^*} = \partial^2 R(u,x)/\partial x^2|_{u=u^*,x=x^*} = (u^*)^2 K[2c'(u^*) + u^*c''(u^*)]/[x^*(1-x^*)]$. Because (u^*,x^*) is an optimal solution, $\partial^2 R(u,x)/\partial x^2|_{u=u^*,x=x^*} \leq 0$; therefore, $\partial u^*(x^*)/\partial x^* > 0$.

Similarly, adding up equations (14) and (15) yields $H(u^*(a), a) \equiv 2 + a - 2u^*K/\Lambda + 2c(u^*) + 2u^*c'(u^*) = 0$. By the Implicit Function Theorem,

$$\partial u^*(a)/\partial a = -\frac{\partial H(u^*,a)/\partial a}{\partial H(u^*,a)/\partial u^*} = -\frac{1}{2[-K/\Lambda + 2c'(u^*) + u^*c''(u^*)]} > 0,$$

where the last inequality is because $2c(u^*) + 2u^*c'(u^*) \leq 0$. Finally, by the chain rule of differentiation, $\partial x^*(a)/\partial a = (\partial u^*(a)/\partial a)/(\partial u^*(x^*)/\partial x^*) > 0$. $\square$

Proof of Corollary 1 If $b = 0$, the objective of (P1) is

$$R^{P1}(\lambda_1,\lambda_2) = \lambda_1 + \lambda_2 - (\lambda_1^2+\lambda_2^2)/\Lambda + (\lambda_1+\lambda_2)c[(\lambda_1+\lambda_2)/K] + \lambda_2 a,$$

whereas the objective of (P3) is

$$R^{P3}(\lambda_1,\lambda_2,x) = \lambda_1 + \lambda_2 - (\lambda_1^2+\lambda_2^2)/\Lambda + \lambda_1 c\{\lambda_1/[(1-x)K]\} + \lambda_2 c[\lambda_1/(xK)] + \lambda_2 a.$$

Consider now an optimal solution to (P3) $\{\lambda_1^*, \lambda_2^*, x^* = \lambda_2^*/(\lambda_1^*+\lambda_2^*)\}$ and notice that $(\lambda_1^*,\lambda_2^*)$ is a feasible solution to (P1) and yields the same revenue. Thus, $\max R^{P1}(\lambda_1,\lambda_2) \geq \max R^{P3}(\lambda_1,\lambda_2,x)$. Likewise, consider an optimal solution to (P1) (ξ_1^*,ξ_2^*) and notice that solution $\{\xi_1^*,\xi_2^*, x^* = \xi_2^*/(\xi_1^*+\xi_2^*)\}$ is a feasible solution to (P3) and yields the same revenue. Thus, $\max R^{P1}(\lambda_1,\lambda_2) \leq \max R^{P3}(\lambda_1,\lambda_2,x)$. The last condition along with $\max R^{P1}(\lambda_1,\lambda_2) \geq \max R^{P3}(\lambda_1,\lambda_2,x)$ jointly imply that $\max R^{P1}(\lambda_1,\lambda_2) = \max R^{P3}(\lambda_1,\lambda_2,x)$. Because both (P1) and (P3) have unique optimal solutions, they must have the same optimal solution. $\square$

Proof of Theorem 1 Note that if $b=0$, the optimal solutions and the revenues are the same with or without capacity allocation, as Corollary 1 suggests. Further, by the Envelope Theorem, when classes do not interact, revenue is not a function of b. On the other hand, when classes interact, $\partial[\max R(\lambda_1,\lambda_2)]/\partial b = \lambda_1^*\lambda_2^*/(\lambda_1^*+\lambda_2^*)\geq 0$. Hence, the result. □

Proof of Theorems 2 In the entire proof we make (implicit) use of the fact that if $b_1=b_2=0$, the optimal solutions and the revenues are the same with and without capacity allocation. Also, note that if $\lambda_1^*=0$ in some region of the $b_1\times b_2$ space, revenue is invariant of b_1, b_2 in that region. Thus, throughout this proof, we focus on the case in which $\lambda_1^*\lambda_2^*>0$ in problem (P2) unless we note otherwise. Consider now stationarity conditions (12) and (13). Because $\lambda_1^*\lambda_2^*>0$, $\mu_2^*=\mu_3^*=0$; subtracting (12) from (13) yields the optimal Lagrange multiplier $g^*=\lambda_2^*-\lambda_1^*-a/[b/(\lambda_1^*+\lambda_2^*)+2/\Lambda]$. To study the behavior of the optimal revenue function wrt b_1, b_2, we need the term in the optimal Lagrange function that depends on b_1, b_2. The relevant term is $L(\lambda_1^*,\lambda_2^*)\equiv b\lambda_1^*\lambda_2^*/(\lambda_1^*+\lambda_2^*)+g^*\{[b_1/(\lambda_1^*+\lambda_2^*)+1/\Lambda]\lambda_2^*-[b_2/(\lambda_1^*+\lambda_2^*)+1/\Lambda]\lambda_1^*-a\}$.

Parts (i) and (iv) of the theorems follow from the fact that that $R(\lambda_1^*,\lambda_2^*)\uparrow b_1$, $R(\lambda_1^*,\lambda_2^*)\uparrow b_2$, which we showed in the proof of part (i) of Proposition 3.

Next we show parts (ii) of both theorems, which require $b_2>0>b_1$. If $b_1\leq a-K/\Lambda$, we know from Lemma 2 that without capacity allocation, an exclusive system is optimal. Because the revenue of an exclusive system can be replicated by allocating capacity $x=1$, allocating capacity can only improve revenue. Hence, part (ii)-a of both theorems. To show parts (ii)-b and (ii)-c of both theorems, it suffices to show the following: 1) $R(\lambda_1^*,\lambda_2^*)\uparrow b_1$, $R(\lambda_1^*,\lambda_2^*)\uparrow b_2$; 2) If $b=0$ and $b_2>0>b_1$, it is optimal to allocate capacity. We have already shown (1) and to show (2), suppose that $b_1=-b_2$. In that case, constraint (4) implies $\lambda_2^*-\lambda_1^*=(a+b_2)\Lambda$; thus, $g^*=\lambda_2^*-\lambda_1^*-a/(2/\Lambda)=(a/2+b_2)\Lambda>0$. Now recall that if $b_1=b_2=0$, revenues are the same with and without capacity allocation, and note that for fixed b, $\partial L/\partial b_2-\partial L/\partial b_1=-g^*<0$. Therefore, if $b=0$ and $b_2>0>b_1$, it is optimal to allocate capacity.

Next we show part (iii) of Theorem 2, which requires $b_1>0>b_2$. If $b_2\leq -K/\Lambda$, or $b_1\leq a-K/\Lambda$ and $b_2>-K/\Lambda$, we know from Lemma 2 that without capacity allocation, an exclusive system is optimal. In that case, as we argued earlier, allocating capacity yields the same or higher revenue. Hence, part (iii)-a. Suppose now that $b_1>a-K/\Lambda$ and $b_2>-K/\Lambda$. If $\lambda_1^*=0$ in problem (P2), the proof for part (iii)-a of the theorem applies.

To prove part (iii)-c of Theorem 2, we will show that the revenue from not allocating capacity strictly increases in b_1-b_2 if $b_1=-b_2$ and $\lambda_1^*>0$ in problem (P2). Recall that if $b_1=-b_2$, constraint (4) implies $\lambda_2^*-\lambda_1^*=(a+b_2)\Lambda\Rightarrow g^*=\lambda_2^*-\lambda_1^*-a/(2/\Lambda)=(a/2+b_2)\Lambda$; thus, $g^*\geq 0$ if $b_2\geq -a/2$. Now recall that if $b_1=b_2=0$, revenues are the same with and without capacity allocation, and note that for fixed b, $\partial L/\partial b_1-\partial L/\partial b_2=g^*\leq 0$, where the last inequality is strict if $b_2>-a/2$. Therefore, if $b=0$, $a/2\geq b_1>a-K/\Lambda$, $b_2\geq -a/2$ and $\lambda_1^*>0$ in problem (P2), it is strictly optimal to not allocate capacity. □