Electronic Companion

In this electronic companion, we provide the proofs of analytical results presented in the main paper. We also present additional technical content that may be useful in understanding the main paper but was deferred to this supplemental document in the interest of space.

EC.1. Proofs of Analytical Results

For the proofs of Propositions 1 and 2, it will be convenient to re-write (5) as follows by distributing the uniformization term $\beta V(\mathbf{w},\mathbf{x})$ and assuming without loss of generality that $\beta = 1$:

$$V(\mathbf{w},\mathbf{x}) = \frac{1}{1+\alpha}\left[\sum_{j\in\mathcal{F}}\mu_j\left((b_j\wedge x_j)\left(r_j + V(\mathbf{w},\mathbf{x}-\mathbf{e}_j)\right) + (b_j - b_j\wedge x_j)V(\mathbf{w},\mathbf{x})\right)\right.$$
$$\left. + \max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})}\left\{\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{F}}\tau_{ij}\left((d_{ij}+f_{ij})V(\mathbf{w}-\mathbf{e}_i,\mathbf{x}+\mathbf{e}_j) + (n_i + n^f - d_{ij} - f_{ij})V(\mathbf{w},\mathbf{x})\right)\right\}\right]. \quad \text{(EC.1)}$$

Proof of Proposition 1. We show that Algorithm 1 provides a solution to (6) by iterating through the **for** loop in lines 7-14 of Algorithm 1. Specifically, we will show that at the end of each iteration of the **for** loop, the currently assigned values of $\{d_{ij}\}$ and $\{f_{ij}\}$ solve

$$\max \sum_{(i,j)\in\{list[1],\ldots,list[k]\}} m_{ij}^{\mathbf{w},\mathbf{x}}\left(d_{ij}+f_{ij}\right) \quad \text{(EC.2)}$$

subject to (2), (3), and (4). That is, among only the location-facility pairs already considered, the assignment is optimal. We proceed by induction over the iterations of the loop. Assume that at the beginning of iteration k, we have $\{d_{ij}\}$ and $\{f_{ij}\}$ which solve

$$\max \sum_{(i,j)\in\{list[1],\ldots,list[k-1]\}} m_{ij}^{\mathbf{w},\mathbf{x}}\left(d_{ij}+f_{ij}\right) \quad \text{(EC.3)}$$

subject to (2), (3), and (4), and that of all possible solutions, the current assignment has the largest possible number of unassigned flexible ambulances. The initialization of $d_{ij} = f_{ij} = 0$ clearly satisfies (EC.3) for $k = 1$, since the list of previously considered facilities would be empty; the initialization is the only feasible assignment. We then expand the summation being maximized to include the pair in $list[k]$. First, note that we cannot increase the objective function by re-assigning any dedicated (resp., flexible) ambulance, which is currently assigned to $list[l]$ for any $l < k$, to $list[k]$, since the list is sorted in decreasing order of the objective coefficient $m_{ij}^{\mathbf{w},\mathbf{x}}$. Now, clearly to solve (EC.2), we want to assign as many ambulances as is feasible to $list[k]$, since future items on $list$ have only the same or lower objective coefficient. Moreover, because the existing solution has, among all optimal solutions, the largest possible number of unassigned flexible ambulances, if we swap any of the currently assigned ambulances in $list[1],\ldots,list[k-1]$ with a currently unassigned ambulance of a different type, we would have to replace a dedicated ambulance with a flexible one, which could not improve (EC.2), since a flexible ambulance can always be assigned to $list[k]$ (which has the largest objective coefficient of the pairs remaining to be considered), but not all dedicated ambulances can be assigned to $list[k]$. Thus, the solution at the end of the **for** loop solves (EC.2). Finally, we conclude by noting that the ordering of lines 8 and 9 of Algorithm 1 ensures that we maintain the property that among all solutions to (EC.2), the one at the end of the **for** loop will have the largest possible number of unassigned flexible ambulances. □

For the proof of Proposition 2, we will make use of a finite-horizon version of the main Markov decision process (MDP).

Definition EC.1 (Finite-horizon MDP). We define $V^t(\mathbf{w},\mathbf{x})$ to be the maximum expected reward earned when the system starts in state $(\mathbf{w},\mathbf{x}) \in \mathcal{S}$ and then runs for $t \geq 1$ additional epochs, each epoch occurring according to a Poisson process with rate β. Then, as in (EC.1), we have

$$V^{t+1}(\mathbf{w},\mathbf{x}) = \frac{1}{1+\alpha}\left[\sum_{j\in\mathcal{F}} \mu_j \left((b_j \wedge x_j)\left(r_j + V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_j)\right) + (b_j - b_j \wedge x_j)V^t(\mathbf{w},\mathbf{x})\right) + \max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})}\left\{\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{F}} \tau_{ij}\left((d_{ij}+f_{ij})V^t(\mathbf{w}-\mathbf{e}_i,\mathbf{x}+\mathbf{e}_j) + (n_i + n^f - d_{ij} - f_{ij})V^t(\mathbf{w},\mathbf{x})\right)\right\}\right].$$

In our proofs, we will use the following abbreviated form of the finite-horizon optimality equation:

$$V^{t+1}(\mathbf{w},\mathbf{x}) = \frac{1}{1+\alpha}\left(r(\mathbf{x}) + c^{t+1}(\mathbf{w},\mathbf{x}) + a^{t+1}(\mathbf{w},\mathbf{x})\right), \tag{EC.4}$$

where $r(\mathbf{x})$ is the instantaneous reward due to service completions,

$$c^{t+1}(\mathbf{w},\mathbf{x}) = \sum_{j\in\mathcal{F}} \mu_j (b_j \wedge x_j) V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_j) + (b_j - b_j\wedge x_j)V^t(\mathbf{w},\mathbf{x})$$

is the marginal expected future reward that will be received if the next transition is a service completion,

$$a^{t+1}(\mathbf{w},\mathbf{x}) = \max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})}\left\{\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{F}} \tau_{ij}\left((d_{ij}+f_{ij})V^t(\mathbf{w}-\mathbf{e}_i,\mathbf{x}+\mathbf{e}_j) + (n_i+n^f-d_{ij}-f_{ij})V^t(\mathbf{w},\mathbf{x})\right)\right\} \tag{EC.5}$$

is the marginal expected future reward due to arrivals to the facilities. Finally, we define $V^0(\mathbf{w},\mathbf{x}) = 0$ for all $(\mathbf{w},\mathbf{x}) \in \mathcal{S}$.

We next state two Lemmas that are used in the proof of Proposition 2.

Lemma EC.1 **(Chapter 2.3, Proposition 3.1, Ross (1983))**. *For any bounded* $V^0(\mathbf{w},\mathbf{x})$, *we have* $V^t(\mathbf{w},\mathbf{x}) \to V(\mathbf{w},\mathbf{x})$ *uniformly as* $t\to\infty$.

Lemma EC.2. *The following inequalities hold for all states* $(\mathbf{w},\mathbf{x}) \in \mathcal{S}$, *for all locations* $i \in \mathcal{L}$, *and for all facilities* $j \in \mathcal{F}$:

$$V(\mathbf{w},\mathbf{x}+\mathbf{e}_j) \geq V(\mathbf{w},\mathbf{x}) \tag{EC.6}$$

$$V(\mathbf{w}+\mathbf{e}_i,\mathbf{x}) \geq V(\mathbf{w},\mathbf{x}) \tag{EC.7}$$

$$V(\mathbf{w},\mathbf{x}) + r_j \geq V(\mathbf{w},\mathbf{x}+\mathbf{e}_j), \tag{EC.8}$$

$\forall (\mathbf{w},\mathbf{x}) \in \mathcal{S}$.

Proof. By Lemma EC.1 and our assumption that $V^0(\mathbf{w},\mathbf{x}) = 0$ for all $(\mathbf{w},\mathbf{x}) \in \mathcal{S}$, it is sufficient to show that the following inequalities hold for all values of $t \geq 0$ and $(\mathbf{w},\mathbf{x}) \in \mathcal{S}$:

$$V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j) \geq V^t(\mathbf{w},\mathbf{x}) \quad \forall j \in \mathcal{F}. \tag{EC.9}$$

$$V^t(\mathbf{w}+\mathbf{e}_i,\mathbf{x}) \geq V^t(\mathbf{w},\mathbf{x}) \quad \forall i \in \mathcal{L}. \tag{EC.10}$$

$$V^t(\mathbf{w},\mathbf{x}) + r_j \geq V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j). \quad \forall j \in \mathcal{F}. \tag{EC.11}$$

We prove (EC.9), (EC.10), and (EC.11) by joint induction on $t \geq 0$. The base cases with $t=0$ for (EC.9), (EC.10), and (EC.11) are trivial because $r_j > 0, \forall j \in \mathcal{F}$, and $V^0(\mathbf{w},\mathbf{x}) = 0$ for all $(\mathbf{w},\mathbf{x}) \in \mathcal{S}$. We now assume that (EC.9), (EC.10), and (EC.11) hold for some $t \geq 0$, and show that the same will hold for $t+1$.

We observe from (EC.4) that we can examine the inductive step for each of the three functions making up $V^{t+1}(\cdot)$ individually. For compactness of notation, define $I_j = 1$ if $x_j < b_j$ and $I_j = 0$ if $x_j \geq b_j$. Then, we have

$$\begin{aligned}
&r(\mathbf{x}+\mathbf{e}_j) + c^{t+1}(\mathbf{w},\mathbf{x}+\mathbf{e}_j) \\
&= r(\mathbf{x}) + \mu_j r_j I_j + \sum_{s\in\mathcal{F}} \mu_s((b_s \wedge x_s)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j-\mathbf{e}_s) + (b_s - b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j)) \\
&\quad + I_j\mu_j(V^t(\mathbf{w},\mathbf{x}) - V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j)) \\
\geq\, & r(\mathbf{x}) + \sum_{s\in\mathcal{F}} \mu_s((b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_s) + (b_s - b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x})) \\
=\, & r(\mathbf{x}) + c^{t+1}(\mathbf{w},\mathbf{x}),
\end{aligned}$$

by first applying (EC.11) at time t for the case where $I_j = 1$, and then directly applying the inductive hypothesis (EC.9) for all terms in the summation.

Let $(\mathbf{d}^*,\mathbf{f}^*)$ be the optimal action in state $(\mathbf{w},\mathbf{x})$ at time t.

$$\begin{aligned}
a^{t+1}(\mathbf{w},\mathbf{x}+\mathbf{e}_j) &= \max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})} \left\{ \sum_{i\in\mathcal{L}}\sum_{s\in\mathcal{F}} \tau_{ij}\left((d_{is}+f_{is})V^t(\mathbf{w}-\mathbf{e}_i,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_s) + (n_i+n^f-d_{is}-f_{is})V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j)\right)\right\} \\
&\geq \sum_{i\in\mathcal{L}}\sum_{s\in\mathcal{F}} \tau_{ij}\left((d^*_{is}+f^*_{is})V^t(\mathbf{w}-\mathbf{e}_i,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_s) + (n_i+n^f-d^*_{is}-f^*_{is})V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j)\right) \\
&\geq \sum_{i\in\mathcal{L}}\sum_{s\in\mathcal{F}} \tau_{ij}\left((d^*_{is}+f^*_{is})V^t(\mathbf{w}-\mathbf{e}_i,\mathbf{x}+\mathbf{e}_s) + (n_i+n^f-d^*_{is}-f^*_{is})V^t(\mathbf{w},\mathbf{x})\right) \\
&= \max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})} \left\{ \sum_{i\in\mathcal{L}}\sum_{s\in\mathcal{F}} \tau_{ij}\left((d_{is}+f_{is})V^t(\mathbf{w}-\mathbf{e}_i,\mathbf{x}+\mathbf{e}_s) + (n_i+n^f-d_{is}-f_{is})V^t(\mathbf{w},\mathbf{x})\right)\right\} \\
&= a^{t+1}(\mathbf{w},\mathbf{x}),
\end{aligned}$$

where the first inequality follows from the fact that $(\mathbf{d}^*,\mathbf{f}^*)$ is a feasible action in state $(\mathbf{w},\mathbf{x}+\mathbf{e}_j)$, and the second inequality follows by applying the inductive hypothesis (EC.9). We conclude that (EC.9) holds for $t+1$.

We now show that (EC.10) holds for $t+1$. Again, we can examine the inductive step for each of the functions making up $V^{t+1}(\cdot)$ individually. First note that we do not need to consider $r(\mathbf{x})$ because it does not depend on $\mathbf{w}$. Next, we have

$$\begin{aligned}
c^{t+1}(\mathbf{w}+\mathbf{e}_i,\mathbf{x}) &= \sum_{s\in\mathcal{F}} \mu_s((b_s\wedge x_s)V^t(\mathbf{w}+\mathbf{e}_i,\mathbf{x}-\mathbf{e}_s) + (b_s - b_s\wedge x_s)V^t(\mathbf{w}+\mathbf{e}_i,\mathbf{x})) \\
&\geq \sum_{s\in\mathcal{F}} \mu_s((b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_s) + (b_s - b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x})) \\
&= c^{t+1}(\mathbf{w},\mathbf{x}),
\end{aligned}$$

by directly applying the inductive hypothesis (EC.10) to each $V^t(\cdot)$.

Let $(\mathbf{d}^*, \mathbf{f}^*)$ be the optimal actions in state $(\mathbf{w}, \mathbf{x})$ at time t, and note that $(\mathbf{d}^*, \mathbf{f}^*) \in \mathcal{A}(\mathbf{w})$ implies $(\mathbf{d}^*, \mathbf{f}^*) \in \mathcal{A}(\mathbf{w}+\mathbf{e}_i)$, because constraint (2) is relaxed. Then, we have

$$\begin{aligned}
&a^{t+1}(\mathbf{w}+\mathbf{e}_i, \mathbf{x}) \\
&= \max_{(\mathbf{d},\mathbf{f}) \in \mathcal{A}(\mathbf{w}+\mathbf{e}_i)} \left\{ \sum_{k\in\mathcal{L}} \sum_{j\in\mathcal{F}} \tau_{kj} \left((d_{kj}+f_{kj}) V^t(\mathbf{w}+\mathbf{e}_i-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_j) + (n_k+n^f-d_{kj}-f_{kj}) V^t(\mathbf{w}+\mathbf{e}_i, \mathbf{x}) \right) \right\} \\
&\geq \sum_{k\in\mathcal{L}} \sum_{j\in\mathcal{F}} \tau_{kj} \left((d^*_{kj}+f^*_{kj}) V^t(\mathbf{w}+\mathbf{e}_i-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_j) + (n_k+n^f-d^*_{kj}-f^*_{kj}) V^t(\mathbf{w}+\mathbf{e}_i, \mathbf{x}) \right) \\
&\geq \sum_{k\in\mathcal{L}} \sum_{j\in\mathcal{F}} \tau_{kj} \left((d^*_{kj}+f^*_{kj}) V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_j) + (n_k+n^f-d^*_{kj}-f^*_{kj}) V^t(\mathbf{w}, \mathbf{x}) \right) \\
&= \max_{(\mathbf{d},\mathbf{f}) \in \mathcal{A}(\mathbf{w})} \left\{ \sum_{k\in\mathcal{L}} \sum_{j\in\mathcal{F}} \tau_{kj} \left((d_{kj}+f_{kj}) V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_j) + (n_k+n^f-d_{kj}-f_{kj}) V^t(\mathbf{w}, \mathbf{x}) \right) \right\} \\
&= a^{t+1}(\mathbf{w}, \mathbf{x}),
\end{aligned}$$

where the first inequality is due to the feasibility of $(\mathbf{d}^*, \mathbf{f}^*)$, and the second inequality is obtained by applying the inductive hypothesis (EC.10) to each $V^t(\mathbf{w}, \mathbf{x})$.

Finally, for (EC.11), we again show the proof in pieces, and by using the assumption that $\beta = 1$, we distribute r_j accordingly among the terms making up $V^t(\mathbf{w}, \mathbf{x})$. We have

$$\begin{aligned}
&r(\mathbf{x}) + c^{t+1}(\mathbf{w}, \mathbf{x}) + \sum_{s\in\mathcal{F}} \mu_s b_s r_j \\
=&\sum_{s\in\mathcal{F}} \mu_s \left[r_s(b_s \wedge x_s) + (b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x}-\mathbf{e}_s) + (b_s - b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x}) + b_s r_j \right] \\
=&\sum_{s\in\mathcal{F}\setminus\{j\}} \mu_s \left[r_s(b_s \wedge x_s) + (b_s \wedge x_s)(V^t(\mathbf{w}, \mathbf{x}-\mathbf{e}_s) + r_j) + (b_s - b_s \wedge x_s)(V^t(\mathbf{w}, \mathbf{x}) + r_j) \right] \\
&+ \mu_j \left[r_j(b_j \wedge x_j) + (b_j \wedge x_j)(V^t(\mathbf{w}, \mathbf{x}-\mathbf{e}_j) + r_j) + (b_j - b_j \wedge (x_j+1))(V^t(\mathbf{w}, \mathbf{x}) + r_j) + I_j(V^t(\mathbf{w}, \mathbf{x}) + r_j) \right] \\
\geq&\sum_{s\in\mathcal{F}\setminus\{j\}} \mu_s \left[r_s(b_s \wedge x_s) + (b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x}+\mathbf{e}_j-\mathbf{e}_s) + (b_s - b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x}+\mathbf{e}_j) \right] \\
&+ \mu_j \left[r_j(b_j \wedge (x_j+1)) + (b_j \wedge (x_j+1)) V^t(\mathbf{w}, \mathbf{x}) + (b_j - b_j \wedge (x_j+1)) V^t(\mathbf{w}, \mathbf{x}+\mathbf{e}_j) \right] \\
=&r(\mathbf{x}+\mathbf{e}_j) + c^{t+1}(\mathbf{w}, \mathbf{x}+\mathbf{e}_j),
\end{aligned} \tag{EC.12}$$

using the inductive hypothesis (EC.11) on every instance of $V^t(\cdot) + r_j$ except for the last, and combining terms. Let $(\mathbf{d}^*, \mathbf{f}^*)$ be the optimal action in state $(\mathbf{w}, \mathbf{x}+\mathbf{e}_j)$ at time t, and note that $\mathcal{A}(\mathbf{w})$ does not depend on $\mathbf{x}$. Then, we have

$$\begin{aligned}
&a^{t+1}(\mathbf{w}, \mathbf{x}) + \sum_{k\in\mathcal{L}} \sum_{l\in\mathcal{F}} \tau_{kl}(n_k+n^f) r_j \\
&= \max_{(\mathbf{d},\mathbf{f}) \in \mathcal{A}(\mathbf{w})} \left\{ \sum_{k\in\mathcal{L}} \sum_{l\in\mathcal{F}} \tau_{kl} \left((d_{kl}+f_{kl})(V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_l) + r_j) + (n_k+n^f-d_{kl}-f_{kl})(V^t(\mathbf{w}, \mathbf{x}) + r_j) \right) \right\} \\
&\geq \sum_{k\in\mathcal{L}} \sum_{l\in\mathcal{F}} \tau_{kl} \left((d^*_{kl}+f^*_{kl})(V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_l) + r_j) + (n_k+n^f-d^*_{kl}-f^*_{kl})(V^t(\mathbf{w}, \mathbf{x}) + r_j) \right) \\
&\geq \sum_{k\in\mathcal{L}} \sum_{l\in\mathcal{F}} \tau_{kl} \left((d^*_{kl}+f^*_{kl}) V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_j+\mathbf{e}_l) + (n_k+n^f-d^*_{kl}-f^*_{kl}) V^t(\mathbf{w}, \mathbf{x}+\mathbf{e}_j) \right) \\
&= \max_{(\mathbf{d},\mathbf{f}) \in \mathcal{A}(\mathbf{w})} \left\{ \sum_{k\in\mathcal{L}} \sum_{l\in\mathcal{F}} \tau_{kl} \left((d_{kl}+f_{kl}) V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_j+\mathbf{e}_l) + (n_k+n^f-d_{kl}-f_{kl}) V^t(\mathbf{w}, \mathbf{x}+\mathbf{e}_j) \right) \right\} \\
&= a^{t+1}(\mathbf{w}, \mathbf{x}+\mathbf{e}_j),
\end{aligned} \tag{EC.13}$$

where the first inequality is due to the feasibility of $(\mathbf{d}^*, \mathbf{f}^*)$, and the second inequality is obtained by applying the inductive hypothesis. Combining (EC.12) and (EC.13) completes the proof. $\square$

Proof of Proposition 2. The result is trivial when $j = l$. Assume that $j, l \in \mathcal{F}$ and $j \neq l$. By Lemma EC.1 and our assumption that $V^0(\mathbf{w}, \mathbf{x}) = 0$ for all $(\mathbf{w}, \mathbf{x}) \in \mathcal{S}$, it is sufficient to show that for any two facilities $i, j \in \mathcal{F}$ such that $r_j \geq r_l$, $\mu_j \geq \mu_l$, $b_j \geq b_l$, and $\tau_{ij} = \tau_{il}$ for all $i \in \mathcal{L}$, if $x_j \leq x_l$, then

$$V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \geq V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_l), \tag{EC.14}$$

$$V^t(\mathbf{w}, T_{jl}\mathbf{x}) \geq V^t(\mathbf{w}, \mathbf{x}), \tag{EC.15}$$

for all $t \geq 0$, where $T_{jl}(\mathbf{x})$ is the linear transformation that swaps the jth element of $\mathbf{x}$ with the lth element of $\mathbf{x}$. We prove (EC.14) and (EC.15) by induction on t. The base case requires that the inequalities hold for $V^0(\cdot)$, which is trivial because $V^0(\cdot)$ is always zero. Suppose that (EC.14) and (EC.15) hold for for some arbitrary $t \geq 0$. We will proceed by showing that when $x_j \leq x_l$, the same inequalities hold for $V^{t+1}(\cdot)$. For $x_j = x_l$, (EC.15) holds trivially, and (EC.14) follows immediately from applying (EC.15); thus, we show the inductive step only for $0 \leq x_j < x_l$.

Inductive step for (EC.14). We break down $V^{t+1}(\cdot)$ into pieces and show the inductive step for each piece. We first show that $r(\mathbf{x} + \mathbf{e}_j) + c^{t+1}(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \geq r(\mathbf{x} + \mathbf{e}_l) + c^{t+1}(\mathbf{w}, \mathbf{x} + \mathbf{e}_l)$. First consider the case where $x_l < b_l$. In this case, because $x_j < x_l$ and $b_l \leq b_j$, we also have $x_j < b_j$ and so

$$\begin{aligned}
&r(\mathbf{x} + \mathbf{e}_j) + c^{t+1}(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \\
=&r(\mathbf{x}) + \mu_j r_j + \sum_{s \neq j,l} \mu_s \left((b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j - \mathbf{e}_s) + (b_s - b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \right) \\
&+ \mu_j (x_j + 1) V^t(\mathbf{w}, \mathbf{x}) + \mu_l x_l V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j - \mathbf{e}_l) + (b_j \mu_j + b_l \mu_l - (x_j + 1)\mu_j - x_l \mu_l) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \\
=&r(\mathbf{x}) + \mu_j (r_j + V^t(\mathbf{w}, \mathbf{x})) + \sum_{s \neq j,l} \mu_s \left((b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j - \mathbf{e}_s) + (b_s - b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \right) \\
&+ \mu_j x_j V^t(\mathbf{w}, \mathbf{x}) + \mu_l x_l V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j - \mathbf{e}_l) + (b_j \mu_j + b_l \mu_l - x_j \mu_j - (x_l + 1)\mu_l) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \\
&- (\mu_j - \mu_l) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \\
\geq&r(\mathbf{x}) + (\mu_j - \mu_l) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) + \mu_l (r_j + V^t(\mathbf{w}, \mathbf{x})) \\
&+ \sum_{s \neq j,l} \mu_s \left((b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_l - \mathbf{e}_s) + (b_s - b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_l) \right) \\
&+ \mu_j x_j V^t(\mathbf{w}, \mathbf{x} - \mathbf{e}_j + \mathbf{e}_l) + \mu_l x_l V^t(\mathbf{w}, \mathbf{x}) + (b_j \mu_j + b_l \mu_l - x_j \mu_j - (x_l + 1)\mu_l) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_l) \\
&- (\mu_j - \mu_l) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_j) \\
\geq&r(\mathbf{x}) + \mu_l r_l \\
&+ \sum_{s \neq j,l} \mu_s \left((b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_l - \mathbf{e}_s) + (b_s - b_s \wedge x_s) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_l) \right) \\
&+ \mu_j x_j V^t(\mathbf{w}, \mathbf{x} - \mathbf{e}_j + \mathbf{e}_l) + \mu_l (x_l + 1) V^t(\mathbf{w}, \mathbf{x}) + (b_j \mu_j + b_l \mu_l - x_j \mu_j - (x_l + 1)\mu_l) V^t(\mathbf{w}, \mathbf{x} + \mathbf{e}_l) \\
=&r(\mathbf{x} + \mathbf{e}_l) + c^{t+1}(\mathbf{w}, \mathbf{x} + \mathbf{e}_l),
\end{aligned}$$

by applying (EC.11) and (EC.14), and using the facts that $\mu_j \geq \mu_l$ and $r_j \geq r_l$. Now, consider the case where $x_l \geq b_l$. We have

$$r(\mathbf{x} + \mathbf{e}_j) + c^{t+1}(\mathbf{w}, \mathbf{x} + \mathbf{e}_j)$$

$$
\begin{aligned}
=&r(\mathbf{x})+\mu_j r_j I_j+\sum_{s\neq j}\mu_s\left((b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j-\mathbf{e}_s)+(b_s-b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j)\right)\\
&+\mu_j((b_j\wedge x_j+I_j)V^t(\mathbf{w},\mathbf{x})+(b_j-b_j\wedge x_j-I_j)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j))\\
\geq&r(\mathbf{x})+\sum_{s\neq j}\mu_s\left((b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_l-\mathbf{e}_s)+(b_s-b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_l)\right)\\
&+\mu_j(b_j\wedge x_j)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_l-\mathbf{e}_j)+(b_j-b_j\wedge x_j)V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_l)+I_j\mu_j(r_j+V^t(\mathbf{w},\mathbf{x})-V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j))\\
\geq&r(\mathbf{x}+\mathbf{e}_l)+c^{t+1}(\mathbf{w},\mathbf{x}+\mathbf{e}_l),
\end{aligned}
$$

where the first inequality follows from (EC.14) and the second inequality follows because $I_j\mu_j(r_j+V^t(\mathbf{w},\mathbf{x})-V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j))\geq 0$ by (EC.11).

Next, let $(\mathbf{d}^*,\mathbf{f}^*)$ be the optimal action in state $(\mathbf{w},\mathbf{x}+\mathbf{e}_l)$ at time t, and note that $\mathcal{A}(\mathbf{w})$ does not depend on $\mathbf{x}$. Then, we have

$$
\begin{aligned}
&a^{t+1}(\mathbf{w},\mathbf{x}+\mathbf{e}_j)\\
&=\max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})}\left\{\sum_{k\in\mathcal{L}}\sum_{s\in\mathcal{F}}\tau_{ks}\left((d_{ks}+f_{ks})V^t(\mathbf{w}-\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_s)+(n_k+n^f-d_{ks}-f_{ks})V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j)\right)\right\}\\
&\geq\sum_{k\in\mathcal{L}}\sum_{s\in\mathcal{F}}\tau_{ks}\left((d^*_{ks}+f^*_{ks})V^t(\mathbf{w}-\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_s)+(n_k+n^f-d^*_{ks}-f^*_{ks})(V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_j)\right)\\
&\geq\sum_{k\in\mathcal{L}}\sum_{s\in\mathcal{F}}\tau_{ks}\left((d^*_{ks}+f^*_{ks})V^t(\mathbf{w}-\mathbf{e}_k,\mathbf{x}+\mathbf{e}_l+\mathbf{e}_s)+(n_k+n^f-d^*_{ks}-f^*_{ks})V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_l)\right)\\
&=\max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})}\left\{\sum_{k\in\mathcal{L}}\sum_{s\in\mathcal{F}}\tau_{ks}\left((d_{ks}+f_{ks})V^t(\mathbf{w}-\mathbf{e}_k,\mathbf{x}+\mathbf{e}_l+\mathbf{e}_s)+(n_k+n^f-d_{ks}-f_{ks})V^t(\mathbf{w},\mathbf{x}+\mathbf{e}_l)\right)\right\}\\
&=a^{t+1}(\mathbf{w},\mathbf{x}+\mathbf{e}_l),
\end{aligned}
$$

where the first inequality is due to the feasibility of $(\mathbf{d}^*,\mathbf{f}^*)$, and the second inequality is obtained by applying the inductive hypothesis (EC.14), including in state $(\mathbf{w},\mathbf{x}+\mathbf{e}_j)$ because we need only to show the inequality for $x_j<x_l$, while we assumed it held for $x_j\leq x_l$.

Inductive step for (EC.15). We break down $V^{t+1}(\cdot)$ into pieces and show the inductive step for each piece. We first show that $r(T_{jl}(\mathbf{x}))+c^{t+1}(\mathbf{w},T_{jl}(\mathbf{x}))\geq r(\mathbf{x})+c^{t+1}(\mathbf{w},\mathbf{x})$. We have

$$
\begin{aligned}
&r(T_{jl}(\mathbf{x}))+c^{t+1}(\mathbf{w},T_{jl}(\mathbf{x}))\\
=&r(\mathbf{x})+\mu_j r_j(b_j\wedge x_l-b_j\wedge x_j)-\mu_l r_l(b_l\wedge x_l-b_l\wedge x_j)\\
&+\sum_{s\neq j,l}\mu_s\left[(b_s\wedge x_s)V^t(\mathbf{w},T_{jl}(\mathbf{x})-\mathbf{e}_s)+(b_s-b_s\wedge x_s)V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right]\\
&+\mu_j\left[(b_j\wedge x_l)V^t(\mathbf{w},T_{jl}(\mathbf{x})-\mathbf{e}_j)+(b_j-b_j\wedge x_l)V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right]\\
&+\mu_l\left[(b_l\wedge x_j)V^t(\mathbf{w},T_{jl}(\mathbf{x})-\mathbf{e}_l)+(b_l-b_l\wedge x_j)V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right]\\
\geq&r(\mathbf{x})+\mu_j r_j(b_j\wedge x_l-b_j\wedge x_j)-\mu_l r_l(b_l\wedge x_l-b_l\wedge x_j)\\
&+\sum_{s\neq j,l}\mu_s\left[(b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_s)+(b_s-b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x})\right]\\
&+\mu_j\left[(b_j\wedge x_l)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)+(b_j-b_j\wedge x_l)V^t(\mathbf{w},\mathbf{x})\right]\\
&+\mu_l\left[(b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_j)+(b_l-b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x})\right]\\
=&r(\mathbf{x})+\mu_j r_j(b_j\wedge x_l-b_j\wedge x_j)-\mu_l r_l(b_l\wedge x_l-b_l\wedge x_j)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s\neq j,l}\mu_s\left[(b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_s)+(b_s-b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x})\right]\\
&+(\mu_j(b_j\wedge x_j)-\mu_l(b_l\wedge x_j))V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)+(\mu_l(b_l\wedge x_j)+\mu_j(b_j\wedge x_l-b_j\wedge x_j))V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)\\
&+\mu_j(b_j-b_j\wedge x_l)V^t(\mathbf{w},\mathbf{x})\\
&+\mu_l(b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_j)\\
&+\mu_l(b_l-b_l\wedge x_l)V^t(\mathbf{w},\mathbf{x})+\mu_l(b_l\wedge x_l-b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x})
\end{aligned}
$$

by applying the inductive hypothesis (EC.15) to each term; which is then

$$
\begin{aligned}
\geq & r(\mathbf{x})+\mu_j r_j(b_j\wedge x_l-b_j\wedge x_j)-\mu_l r_l(b_l\wedge x_l-b_l\wedge x_j)\\
&+\sum_{s\neq j,l}\mu_s\left[(b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_s)+(b_s-b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x})\right]\\
&+(\mu_j(b_j\wedge x_j)-\mu_l(b_l\wedge x_j))V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_j)+(\mu_l(b_l\wedge x_j)+\mu_j(b_j\wedge x_l-b_j\wedge x_j))V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)\\
&+\mu_j(b_j-b_j\wedge x_l)V^t(\mathbf{w},\mathbf{x})+\mu_l(b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_j)\\
&+\mu_l(b_l-b_l\wedge x_l)V^t(\mathbf{w},\mathbf{x})+\mu_l(b_l\wedge x_l-b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x})
\end{aligned}
$$

by applying the inductive hypothesis (EC.14) once. Rearranging terms, the above result is therefore

$$
\begin{aligned}
=& r(\mathbf{x})+\mu_j(r_j-r_l)(b_j\wedge x_l-b_j\wedge x_j)\\
&+\sum_{s\neq j,l}\mu_s\left[(b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_s)+(b_s-b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x})\right]\\
&+\mu_j(b_j\wedge x_j)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_j)\\
&+\mu_j(b_j-b_j\wedge x_l)V^t(\mathbf{w},\mathbf{x})+\mu_l(b_l\wedge x_l-b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x})\\
&+(\mu_j(b_j\wedge x_l-b_j\wedge x_j)-\mu_l(b_l\wedge x_l-b_l\wedge x_j))(V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)+r_l)\\
&+\mu_l(b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)+\mu_l(b_l\wedge x_l-b_l\wedge x_j)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)+\mu_l(b_l-b_l\wedge x_l)V^t(\mathbf{w},\mathbf{x})\\
\geq & r(\mathbf{x})+\sum_{s\neq j,l}\mu_s\left[(b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_s)+(b_s-b_s\wedge x_s)V^t(\mathbf{w},\mathbf{x})\right]\\
&+\mu_j(b_j\wedge x_j)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_j)+\mu_j(b_j-b_j\wedge x_j)V^t(\mathbf{w},\mathbf{x})\\
&+\mu_l(b_l\wedge x_l)V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)+\mu_l(b_l-b_l\wedge x_l)V^t(\mathbf{w},\mathbf{x})\\
=& r(\mathbf{x})+c^{t+1}(\mathbf{w},\mathbf{x}),
\end{aligned}
$$

by applying (EC.11), which shows that $V^t(\mathbf{w},\mathbf{x}-\mathbf{e}_l)+r_l\geq V^t(\mathbf{w},\mathbf{x})$, and using the facts that $r_j\geq r_l$, $x_j<x_l$, $\mu_j\geq\mu_l$, and $b_j\geq b_l$, and combining terms.

Next, we must show that $a^{t+1}(\mathbf{w},T_{jl}(\mathbf{x}))\geq a^{t+1}(\mathbf{w},\mathbf{x})$. Let $(\mathbf{d}^*,\mathbf{f}^*)$ be the optimal action in state $(\mathbf{w},\mathbf{x})$ at time t, and note that $\mathcal{A}(\mathbf{w})$ does not depend on $\mathbf{x}$. Now, set $\tilde{d}_{ks}=d^*_{ks}$ for all $s\neq j,l$, while setting $\tilde{d}_{kl}=d^*_{kj}$ and $\tilde{d}_{kj}=d^*_{kl}$, as well as $\tilde{f}_{ks}=f^*_{ks}$ for all $s\neq j,l$, while setting $\tilde{f}_{kl}=f^*_{kj}$ and $\tilde{f}_{kj}=f^*_{kl}$. Note that $(\tilde{\mathbf{d}},\tilde{\mathbf{f}})$ is a feasible action for state $(\mathbf{w},\mathbf{x})$. Because the feasibility of an action depends only on $\mathbf{w}$, and not on $\mathbf{x}$, it is the case that $(\tilde{\mathbf{d}},\tilde{\mathbf{f}})$ is also a feasible action in state $(\mathbf{w},T_{jl}(\mathbf{x}))$. Then we have

$$
\begin{aligned}
&a^{t+1}(\mathbf{w},T_{jl}(\mathbf{x}))\\
&=\max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})}\left\{\sum_{k\in\mathcal{L}}\sum_{s\in\mathcal{F}}\tau_{ks}\left((d_{ks}+f_{ks})V^t(\mathbf{w}-\mathbf{e}_k,T_{jl}(\mathbf{x})+\mathbf{e}_s)+(n_k+n^f-d_{ks}-f_{ks})V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right)\right\}
\end{aligned}
$$

$$\begin{aligned}
&\geq \sum_{k\in\mathcal{L}}\sum_{s\in\mathcal{F}} \tau_{ks}\left((\tilde{d}_{ks}+\tilde{f}_{ks})V^t(\mathbf{w}-\mathbf{e}_k, T_{jl}(\mathbf{x})+\mathbf{e}_s)+(n_k+n^f-\tilde{d}_{ks}-\tilde{f}_{ks})V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right)\\
&\geq \sum_{k\in\mathcal{L}}\sum_{s\in\mathcal{F}} \tau_{ks}\left((d^*_{ks}+f^*_{ks})V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_s)+(n_k+n^f-d^*_{ks}-f^*_{ks})V^t(\mathbf{w},\mathbf{x})\right)\\
&= \max_{(\mathbf{d},\mathbf{f})\in\mathcal{A}(\mathbf{w})}\left\{\sum_{k\in\mathcal{L}}\sum_{s\in\mathcal{F}} \tau_{ks}\left((d_{ks}+f_{ks})V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_s)+(n_k+n^f-d_{ks}-f_{ks})V^t(\mathbf{w},\mathbf{x})\right)\right\}\\
&= a^{t+1}(\mathbf{w},\mathbf{x}+\mathbf{e}_l),
\end{aligned}$$

where the first inequality is due to the feasibility of $(\tilde{\mathbf{d}},\tilde{\mathbf{f}})$. To obtain the second inequality, for all $s\neq j,l$, we can apply the inductive hypothesis (EC.15) directly. For $s\in\{j,l\}$, we have

$$\begin{aligned}
&\sum_{k\in\mathcal{L}}\left(\tau_{kj}\left((\tilde{d}_{kj}+\tilde{f}_{kj})V^t(\mathbf{w}-\mathbf{e}_k, T_{jl}(\mathbf{x})+\mathbf{e}_j)+(n_k+n^f-\tilde{d}_{kj}-\tilde{f}_{kj})V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right)\right.\\
&\left.\quad+\ \tau_{kl}\left((\tilde{d}_{kl}+\tilde{f}_{kl})V^t(\mathbf{w}-\mathbf{e}_k, T_{jl}(\mathbf{x})+\mathbf{e}_l)+(n_k+n^f-\tilde{d}_{kl}-\tilde{f}_{kl})V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right)\right)\\
=&\sum_{k\in\mathcal{L}}\left(\tau_{kl}\left((d^*_{kl}+f^*_{kl})V^t(\mathbf{w}-\mathbf{e}_k, T_{jl}(\mathbf{x})+\mathbf{e}_j)+(n_k+n^f-d^*_{kl}-f^*_{kl})V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right)\right.\\
&\left.\quad+\ \tau_{kj}\left((d^*_{kj}+f^*_{kj})V^t(\mathbf{w}-\mathbf{e}_k, T_{jl}(\mathbf{x})+\mathbf{e}_l)+(n_k+n^f-d^*_{kj}-f^*_{kj})V^t(\mathbf{w},T_{jl}(\mathbf{x}))\right)\right)
\end{aligned}$$

by the fact that $\tau_{kj}=\tau_{kl}$ and the definition of $(\tilde{\mathbf{d}},\tilde{\mathbf{f}})$. Applying the inductive hypothesis (EC.15), the above expression can be lower-bounded by

$$\begin{aligned}
&\sum_{k\in\mathcal{L}}\left(\tau_{kl}\left((d^*_{kl}+f^*_{kl})V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_l)+(n_k+n^f-d^*_{kl}-f^*_{kl})V^t(\mathbf{w},\mathbf{x})\right)\right.\\
&\left.\quad+\ \tau_{kj}\left((d^*_{kj}+f^*_{kj})V^t(\mathbf{w}-\mathbf{e}_k, \mathbf{x}+\mathbf{e}_j)+(n_k+n^f-d^*_{kj}-f^*_{kj})V^t(\mathbf{w},\mathbf{x})\right)\right)\\
&= a^{t+1}(\mathbf{w},\mathbf{x}),
\end{aligned}$$

which completes the proof. □

Proof of Proposition 3. In the ample capacity case, we have assumed that $b_j\wedge x_j=x_j$ for all possible x_j, $j\in\mathcal{F}$. (For the purposes of uniformization, b_j is required to be finite for all $j\in\mathcal{F}$. Note that there exists b_j satisfying $b_j\wedge x_j=x_j$ for all possible x_j such that $b_j\leq\sum_i w_i$ plus the initial number of patients at facility j.) There are two immediate implications of this assumption important to this proof: (i) all casualties at the facilities are always in service, meaning each casualty in service at facility $j\in\mathcal{F}$ has an expected discounted reward of $r_j\mu_j/(\mu_j+\alpha)$, and (ii) any casualty arriving to facility $j\in S$ immediately enters service. Therefore, for any $(\mathbf{w},\mathbf{s})\in\mathcal{S}$ we have

$$V(\mathbf{w},\mathbf{x})=V(\mathbf{w},0)+\sum_{j\in\mathcal{F}}x_jr_j\left(\frac{\mu_j}{\mu_j+\alpha}\right). \tag{EC.16}$$

Using (EC.16) and defining $\kappa_i(\mathbf{w})=V(\mathbf{w},0)-V(\mathbf{w}-\mathbf{e}_i,0)$ (which is non-negative by (EC.7), we have

$$\begin{aligned}
m_{ij}^{\mathbf{w},\mathbf{x}}&=\tau_{ij}\left(V(\mathbf{w}-\mathbf{e}_i,\mathbf{x}+\mathbf{e}_j)-V(\mathbf{w},\mathbf{x})\right)\\
&=\tau_{ij}\left(V(\mathbf{w}-\mathbf{e}_i,0)-V(\mathbf{w},0)+\frac{r_j\mu_j}{\mu_j+\alpha}\right)\\
&=\tau_{ij}\left(\frac{r_j\mu_j}{\mu_j+\alpha}-\kappa_i(\mathbf{w})\right).
\end{aligned}$$

We prove the result that $\kappa_i(\mathbf{w})$ diminishes exponentially fast in w_i by a coupling argument. Consider two coupled processes: process 1 starts with state $(\mathbf{w},0)$ and takes an optimal action at each decision epoch, while process 2 starts with state $(\mathbf{w}-\mathbf{e}_i,0)$ and takes the same action as process 1 at each decision epoch, except that if the action taken by process 1 is for an ambulance to transport the last casualty from location i (and hence there is no casualty remaining at location i in process 2), the ambulance in process 2 idles. The expected total discounted reward earned by process 1 is $V(\mathbf{w},0)$. Denote by $\tilde{V}$ the expected total discounted reward earned by process 2. It must be that $\tilde{V} \le V(\mathbf{w}-\mathbf{e}_i,0)$, since process 2 does not necessarily use an optimal policy. Then, $V(\mathbf{w},0)-V(\mathbf{w}-\mathbf{e}_i,0) \le V(\mathbf{w},0)-\tilde{V}$, and therefore, we can bound $\kappa_i(\mathbf{w}) = V(\mathbf{w},0)-V(\mathbf{w}-\mathbf{e}_i,0)$ (which is non-negative by (EC.7)) from above by considering the difference in expected total discounted reward between the two coupled processes.

Since the two processes are coupled as above, $V(\mathbf{w},0)-\tilde{V} = r_j E[e^{-\alpha T+S_j}]$, where j is the facility where the last casualty at location i is taken in process 1, T is the time when that casualty reaches facility j, and S_j is its service time. Since T and S_j are independent, we have

$$
\begin{aligned}
V(\mathbf{w},0)-\tilde{V} &= r_j E[e^{-\alpha(T+S_j)}] \\
&= E[e^{-\alpha T}]E[r_j e^{-\alpha S_j}] \\
&\le E[e^{-\alpha X_i}]E[r_j e^{-\alpha S_j}],
\end{aligned}
$$

where X_i is an Erlang random variable with w_i phases and rate $(n_i+n^f)\max_{l\in\mathcal{F}}\tau_{il}$. Finally, by choosing j to create an upper bound, we can bound $E[r_j e^{-\alpha S_j}]$ from above by $\max_{j\in\mathcal{F}}\left\{\frac{r_j\mu_j}{\mu_j+\alpha}\right\}$, which completes the proof. □

Proof of Proposition 4. Since the decisions in the static policy are random, the expected total discounted reward is separable by facility. That is, we can write $V_\infty^\gamma(\mathbf{x}) = \sum_{j\in S} W_j^\gamma(x_j)$, where $W_j^\gamma(x_j)$ is the expected total discounted reward for a specific facility $j\in\mathcal{F}$ with arrivals at rate λ_j and departures at rate $b_j\mu_j$, when there are x_j casualties waiting. For ease of exposition, we suppress the subscript j (corresponding to the facility) and the superscript γ (corresponding to the static policy) everywhere they appear in this proof, and we assume without loss of generality that $b=1$ in this proof. By uniformizing the process corresponding to the facility with uniformization constant $\lambda+\mu$, we can study the embedded discrete-time Markov chain by observing the queue at the facility only at transitions, i.e., arrivals and service completions. Arrivals occur with probability $\lambda/(\lambda+\mu)$ and service completions occur with probability $\mu/(\lambda+\mu)$. By incorporating the discount factor α, we can define $W(x)$ according to the following recursion:

$$
W(x) = \frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\mu}W(x+1) + \frac{\mu}{\lambda+\mu}\left(r+W(x-1)\right)\right),
$$

for $x\ge 1$, or in other words,

$$
\lambda W(x) - (\lambda+\mu+\alpha)W(x-1) + \mu W(x-2) = -\mu r, \tag{EC.17}
$$

for $x\ge 2$. The boundary condition is

$$
\lambda W(1) - (\lambda+\alpha)W(0) = 0, \tag{EC.18}
$$

because there are no service completions when there are zero casualties at the facility.

By using the standard method for solving a nonhomogeneous difference equation with a boundary condition, we obtained

$$W(x) = \frac{\mu r}{\alpha} - \frac{2\mu r}{\lambda + \alpha - \mu + \sqrt{(\lambda+\mu+\alpha)^2 - 4\lambda\mu}} \left(\frac{\lambda+\mu+\alpha - \sqrt{(\lambda+\mu+\alpha)^2-4\lambda\mu}}{2\lambda} \right)^x. \tag{EC.19}$$

Using algebraic manipulation it is straightforward to show that if $W(x)$ is given by the expression in (EC.19), then it satisfies (EC.17) and (EC.18). By summing the above expression over all facilities, we obtain

$$V_\infty^\gamma(\mathbf{x}) = \sum_{j\in\mathcal{F}} \left[\frac{b_j\mu_j r_j}{\alpha} - \frac{2b_j\mu_j r_j}{\lambda_j + \alpha - b_j\mu_j + \eta_j} \left(\frac{\lambda_j + b_j\mu_j + \alpha - \eta_j}{2\lambda_j} \right)^{x_j} \right], \tag{EC.20}$$

which yields (10). □

EC.2. Bernoulli Splitting Policies for Policy Improvement Heuristic

In the following, define Γ to be the set of all possible Bernoulli-splitting policies.

EC.2.1. Optimal Bernoulli-splitting policy

Because applying a step of the policy improvement algorithm results in a dynamic policy that performs at least as well as the initial static policy, intuitively we would expect that the PIH should work well if we initialize it with the optimal Bernoulli-splitting policy, i.e., the policy that yields the largest expected total discounted reward within Γ. Finding this optimal static policy (denoted by ST-O) within Γ requires maximizing $V_\infty^\gamma(\mathbf{x})$, which is given by (EC.20), subject to $\sum_{j\in\mathcal{F}} \rho_{ij} = 1, \forall i \in \mathcal{L}$, $\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{F}} \theta_{ij} = 1$, and $\rho_{ij} \geq 0, \theta_{ij} \geq 0, \forall i \in \mathcal{L}$ and $j \in \mathcal{F}$. Unfortunately, the objective function is nonlinear in ρ_{ij} and θ_{ij}, and hence the problem is difficult to solve except for cases with a small number of locations and facilities, where it can be solved numerically. In our numerical experiments presented in Section EC.3, we report the performance of the policy improvement heuristic with optimal Bernoulli-splitting policy (denoted by PIH-O) for cases with a small number of locations and facilities.

EC.2.2. Fluid approximation

As an alternative to determining the optimal static policy for the infinite-casualty model, we consider finding the optimal static policy for a fluid approximation of the problem. A fluid approximation is one in which the discrete entities are treated as continuous, and transportation of casualties to the facilities and subsequent service thereof is equivalent to the flow of a fluid. Recall that the inverse of τ_{ij} is defined to be the mean travel time to facility j from location i. Therefore, in a fluid approximation, the fluid flowing from location i to facility j first arrives at time $1/\tau_{ij}$, flows at rate $(\rho_{ij}n_i + \theta_{ij}n^f)\tau_{ij}$ units of fluid per time unit, and continuously earns a reward of r_j per unit of fluid. When this reward is discounted to time zero at discount factor α, the total reward earned by fluid flowing from location i to facility j becomes $\int_{1/\tau_{ij}}^\infty (\rho_{ij}n_i + \theta_{ij}n^f)\tau_{ij}r_j e^{-\alpha t}dt = (\rho_{ij}n_i + \theta_{ij}n^f)\tau_{ij}r_j e^{-\frac{\alpha}{\tau_{ij}}}/\alpha$, assuming that the facilities are initially empty and idle. One can then find the optimal static policy for the fluid approximation (denoted by ST-F) by solving the following linear program:

$$\max \quad \sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{F}} \tau_{ij}r_j e^{-\frac{\alpha}{\tau_{ij}}}(\rho_{ij}n_i + \theta_{ij}n^f) \tag{EC.21}$$

$$\text{s.t.} \quad \sum_{i\in\mathcal{L}}(\rho_{ij}n_i+\theta_{ij}n^f)\tau_{ij}\le b_j\mu_j, \quad \forall j\in\mathcal{F} \tag{EC.22}$$

$$\sum_{j\in\mathcal{F}}\rho_{ij}\le 1, \quad \forall i\in\mathcal{L} \tag{EC.23}$$

$$\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{F}}\theta_{ij}\le 1 \tag{EC.24}$$

$$\rho_{ij}\ge 0, \quad \forall i\in\mathcal{L}, j\in\mathcal{F} \tag{EC.25}$$

$$\theta_{ij}\ge 0, \quad \forall i\in\mathcal{L}, j\in\mathcal{F} \tag{EC.26}$$

Constraint set (EC.22) guarantees that the fluid does not arrive to facility j at a total rate faster than $b_j\mu_j$. A solution where fluid arrives at a total rate faster than $b_j\mu_j$ would be feasible, but it could not increase the expected total discounted reward because the fluid can never flow out of facility j (and thus earn a reward) at a rate faster than $b_j\mu_j$. In constraint sets (EC.23)-(EC.24), we allow the total probability for a given location to be less than one and the total probability for the flexible ambulances to be less than one (versus setting them equal to one) in both cases to guarantee that (EC.21) has a feasible solution in cases where transportation resources have a high capacity relative to treatment facilities. Because arrivals to a facility at a rate faster than the facility's service rate would have no benefit in the fluid formulation, it is possible that in the optimal solution to (EC.21), not all transportation capacity is used. In our numerical experiments, we will report the performance of the PIH based on the exact solution to (EC.21) (denoted by PIH-F).

EC.2.3. Greedy assignment for the fluid approximation

Formulation (EC.21) is a continuous relaxation of the generalized assignment problem (see, e.g., Öncan 2007). If the problem is relaxed by removing constraints (EC.22), which would be a good approximation when facilities are fast compared to transportation resources, then the resulting relaxation could be solved optimally via greedy choice. One can also use the idea of greedy choice to obtain a feasible solution to (EC.21) as follows. Initialize by setting $\rho_{ij}=0$ and $\theta_{ij}=0$ for all $i\in\mathcal{L}, j\in\mathcal{F}$. Next, make a list of all facility-location pairs $(i,j), i\in\mathcal{L}, j\in\mathcal{F}$ in decreasing order of $\tau_{ij}r_je^{-\frac{\alpha}{\tau_{ij}}}$. Then, proceeding in order through the list, (i) assign the largest possible non-negative value to ρ_{ij} such that constraints (EC.22) and (EC.23) are satisfied; subsequently (ii) assign the largest possible non-negative value to θ_{ij} such that constraints (EC.22) and (EC.24) are satisfied. Stop when each facility-location pair in the list has been considered once. We call this procedure Algorithm 2, which is shown to return a feasible solution to (EC.21) in Proposition EC.1. We denote the resulting "greedy" policy by ST-G. In our numerical experiments, we will also report the results of the PIH using the probability assignment based on Algorithm 2 (denoted by PIH-G).

PROPOSITION EC.1. *Algorithm 2 returns a feasible solution to* (EC.21).

Proof. We will prove the proposition by showing that at the end of each iteration through the second **for** loop, constraints (EC.22)–(EC.26) are satisfied. When the **for** loop is initialized for $\kappa=1$, $\rho_{ij}=\theta_{ij}=0$ for all $i\in\mathcal{L}, j\in\mathcal{F}$ and thus all constraints are clearly satisfied at the beginning of the first iteration.

Now, we show that if the constraints are satisfied at the beginning of any iteration of the **for** loop, then they will be satisfied at the end of the same iteration. During the first statement of the **for** loop iteration,

Algorithm 2 Greedy algorithm for obtaining a feasible solution to (EC.21).

function GREEDY($\{\tau_{ij}\}, \{b_j\}, \{\mu_j\}, \{r_j\}, \alpha$)
  for all $i \in \mathcal{L}, j \in \mathcal{F}$ **do**
    $\rho_{ij} \leftarrow 0$
    $\theta_{ij} \leftarrow 0$
  $list \leftarrow \{(i,j), i \in \mathcal{L}, j \in \mathcal{F}\}$
  SortDescending($list, \tau_{ij} r_j \exp(-\alpha/\tau_{ij})$)
  for $\kappa = 1$ to Length($list$) **do**
    $(i,j) \leftarrow list[\kappa]$
    $\rho_{ij} \leftarrow \min\left\{1 - \sum_{l\in\mathcal{F}} \rho_{il}, \left(b_j\mu_j - \sum_{k\in\mathcal{L}} \tau_{kj}(\rho_{kj} n_k + \theta_{kj} n^f)\right) / (\tau_{ij} n_i)\right\}$
    $\theta_{ij} \leftarrow \min\left\{1 - \sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{F}} \theta_{kl}, \left(b_j\mu_j - \sum_{k\in\mathcal{L}} \tau_{kj}(\rho_{kj} n_k + \theta_{kj} n^f)\right) / (\tau_{ij} n^f)\right\}$
  return $\{\rho_{ij}, \theta_{ij}\}$

only one ρ_{ij} can be changed, namely the one in the κth position of the list. Since (EC.22) and (EC.23) are satisfied at the beginning of the iteration, both $1 - \sum_{l\in\mathcal{F}} \rho_{il}$ and $\left(b_j\mu_j - \sum_{k\in\mathcal{L}} \tau_{kj}(\rho_{kj} n_k + \theta_{kj} n^f)\right) / (\tau_{ij} n_i)$ are non-negative, and hence ρ_{ij} will also be non-negative at the end of the iteration, satisfying (EC.25). Moreover, because the new value assigned to ρ_{ij} is at most $1 - \sum_{l\in\mathcal{F}} \rho_{il}$, we maintain $\sum_{l\in\mathcal{F}} \rho_{il} \le 1$ and (EC.23) is satisfied at the end of the first statement. An identical argument can be made for (EC.22). During the second statement in the **for** loop, only one θ_{ij} can be changed. We just proved that (EC.22) will be satisfied after the execution of the first statement, and from the beginning of the iteration, we know that (EC.24) was satisfied. Therefore, both $1 - \sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{F}} \theta_{kl}$ and $\left(b_j\mu_j - \sum_{k\in\mathcal{L}} \tau_{kj}(\rho_{kj} n_k + \theta_{kj} n^f)\right) / (\tau_{ij} n^f)$ will be non-negative. Since θ_{ij} is the minimum of these two values, (EC.26) will be satisfied at the end of the iteration. Because the new value assigned to θ_{ij} is at most $1 - \sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{F}} \theta_{kl}$, we maintain $\sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{F}} \theta_{kl} \le 1$ and (EC.24) is satisfied at the end of the iteration. An identical argument can be made for (EC.22).

We have therefore proved that if constraints (EC.22)–(EC.26) are satisfied at the beginning of any iteration of the **for** loop, then they will also be satisfied at the end of such iteration. Since all constraints were satisfied during the initialization step, we have therefore shown that all constraints will be satisfied after every iteration of the **for** loop. Thus, the values $\{\rho_{ij}, \theta_{ij}\}$ returned by Algorithm 2 are feasible. □

EC.3. Numerical experiments: comparison with the optimal policy

Due to the practical difficulty of solving for the optimal policy numerically with a large state space, we evaluate the performance of our heuristics in the limiting regime given by the MDP in (9). In order to test the performance of our heuristics as a solution to the optimization problem (9), we conducted a set of numerical experiments with 1,000 randomly generated instances, each with two locations and two facilities. Using a small number of locations and facilities enables us to find the optimal solution by solving (9) numerically. Each random instance has total service rates ($b_j\mu_j$) and transportation rates (τ_{ij}) chosen uniformly in $[1,10]$. All rewards are set to one, which means that the performance measure is the expected discounted throughput.

Table EC.1 Performance of heuristic policies as a percentage of optimal discounted throughput for the MDP (9).

α	Policy	Min	Q1	Q2	Q3	Max*	Mean
0.1	ST-F	51	78	85	88	91	81
	ST-G	48	75	85	88	91	81
	ST-O	51	79	92	97	99	87
	Myopic	30	94	99	100	100	92
	PIH-F	89	100	100	100	102	99
	PIH-G	95	100	100	100	102	100
	PIH-O	95	99	100	100	102	99
0.7	ST-F	49	66	72	75	80	70
	ST-G	46	65	71	75	80	69
	ST-O	52	72	81	87	94	79
	Myopic	86	97	99	100	100	98
	PIH-F	87	99	100	100	102	99
	PIH-G	91	100	100	100	102	100
	PIH-O	91	98	100	100	102	99
2.0	ST-F	33	59	63	67	73	62
	ST-G	33	57	63	66	73	61
	ST-O	51	66	73	78	87	72
	Myopic	92	98	99	100	100	98
	PIH-F	88	99	100	100	102	99
	PIH-G	90	99	100	100	102	99
	PIH-O	90	98	100	100	102	99

* Because we used state-space truncation to find the optimal policy, some of the heuristics performed better than the so-called optimal solution in certain cases.

For each instance, we calculated the optimal policy using the value iteration algorithm under a truncated state space where each facility had room for 25 casualties to wait for service. We then calculated the expected discounted throughputs for the optimal solution to (9), myopic policy, and the three variations of the policy improvement heuristic, i.e., PIH-F, PIH-G, and PIH-O, assuming that all stations are initially empty and all transportation resources are idle. For comparison, we also computed the performance of the three static policies discussed in Section EC.2, namely, ST-F, ST-G, and ST-O policies. We repeated each experiment with $\alpha \in \{0.1, 0.7, 2.0\}$. The discount factor $\alpha = 0.7$ corresponds to a discount of approximately 50% of the reward per unit time. The other discount factors $\alpha = 0.1$ and $\alpha = 2.0$ are chosen to provide a comparison under (relatively) light and heavy discounting to represent less or more urgent conditions, respectively.

The results from these numerical experiments are shown in Table EC.1. The entries in the third through eighth columns summarize the distribution of the percentage of the optimal discounted throughput attained by each heuristic policy based on 1,000 instances. Specifically, entries in the third through eight columns provide the minimum performance (Min), first quartile (Q1), median (Q2), third quartile (Q3), maximum (Max), and mean in the given order.

Table EC.1 shows that all dynamic heuristics perform well in terms of the mean performance, achieving an average of at least 92% of the value of the optimal policy. However, there is a clear tradeoff between simplicity and performance when we consider the worst-case performances. More specifically, the myopic policy has the simplest index to calculate, but performs poorly in a small number of instances when α is small. On the other hand, PIH performs very well under all three initial static policies for all performance statistics, being essentially indistinguishable from the optimal dynamic policy in the mean performance. It is especially good that starting from the greedy or fluid optimal policies provide a similar performance compared with starting from the optimal static policy, because calculating the optimal static policy is practical only for problems with a relatively small number of locations and stations. Static policies do not come close to the performance of dynamic policies in most cases but appear to perform notably better for smaller α.

EC.4. Simulation study: intermittent state information

We repeated the randomized study discussed in Section 5.3 when state information updates about the number of casualties at each facility arrive according to a Poisson process with a rate of four updates per hour (i.e., on average every 15 minutes), two updates per hour, and no updates, using the modified heuristics developed in Section 4.3.

From equations (11)–(12) and the approximation that $\psi_j(s)$ is a Poisson random variable, we can use generating functions to rewrite the heuristic indices for both the Myopic and PIH policies in the form

$$\tau_{ij} r_j \left[\upsilon_1 F(x_j + y_j; \nu) + \upsilon_2 (1 - F(x_j + y_j; s b_j \mu_j)) \right], \tag{EC.27}$$

where $F(\cdot, \xi)$ denotes the cumulative distribution function of a Poisson random variable with mean $\xi \geq 0$, and the values of υ_1, υ_2, and ν under the Myopic and PIH policies are given in Table EC.2.

Table EC.2 Specific parameter values for Equation (EC.27) under Myopic and PIH policies.

	υ_1	υ_2	ν
Myopic	$e^{s\alpha} \left(\frac{b_j \mu_j}{b_j \mu_j + \alpha} \right)^{x_j + y_j + 1}$	$\frac{b_j \mu_j}{b_j \mu_j + \alpha}$	$s(b_j \mu_j + \alpha)$
PIH	$\frac{b_j \mu_j \exp\left(\frac{s b_j \mu_j (\lambda_j + \eta_j - b_j \mu_j - \alpha)}{b_j \mu_j + \lambda_j + \alpha - \eta_j} \right) \left(\frac{b_j \mu_j + \lambda_j + \alpha - \eta_j}{2\lambda_j} \right)^{x_j + y_j}}{\lambda_j \left(\frac{b_j \mu_j - \lambda_j - \alpha - \eta_j}{b_j \mu_j - \lambda_j + \alpha - \eta_j} \right)}$	$\left(\frac{b_j \mu_j}{\lambda_j} \right) \left(\frac{b_j \mu_j - \lambda_j + \alpha - \eta_j}{b_j \mu_j - \lambda_j - \alpha - \eta_j} \right)$	$\frac{2 s \lambda_j b_j \mu_j}{b_j \mu_j + \lambda_j + \alpha - \eta_j}$

To simulate intermittent state updates, we maintained three state variables in the discrete-event simulator: the actual number of patients at each facility (used internally in the simulation); the observed number of patients at each facility (which was set to the true value at each update epoch); and the number of patients sent to each facility from each location since the last update. Critically, we assumed that each location would know only the number of patients sent *from that location* since the last update, in keeping with our goal of demonstrating the value of dynamic heuristics even when information about the state in other locations is unknown. The state information updates were simulated in the same way for all three dynamic heuristics, including the BLD policy (to be specific, for BLD, we chose the queue with shortest expected time to service completion $\tau_{ij}^{-1} + \mathsf{E}\left[(x_j + 1 + y_j - \psi_j(s) - b_j)^+ / (b_j \mu_j)\right] + \mu_{ij}^{-1}$, where x_j, y_j, and $\psi_j(s)$ are defined in Section 4.3). Results of this study are presented in Tables EC.3, EC.4, and 2. We noted that while performance improvement over BLS (which does not use state information) was somewhat smaller as the frequency of updates decreased, both modified PIH and Myopic performed much better against modified BLD as state updates became less frequent. This result suggests that a heuristic approach to casualty distribution can still perform well without real-time state information.

Table EC.3 Simulation results for modified heuristics when information updates occur with rate 4 per hour.

Trauma Type	No. of Locations	Policy	vs. Baseline Static Min.	Q1	Med.	Q3	Max	# Sig	vs. Baseline Dynamic Min.	Q1	Med.	Q3	Max.	# Sig
Blunt	3	PIH	-10%	37%	47%	69%	233%	297	-20%	2%	4%	8%	23%	255
		Myopic	-1%	29%	40%	64%	215%	298	-38%	-2%	1%	3%	19%	154
	4	PIH	0%	37%	47%	62%	241%	298	-10%	2%	4%	7%	18%	278
		Myopic	0%	30%	41%	55%	230%	300	-25%	-1%	1%	3%	12%	187
	5	PIH	-2%	15%	45%	56%	241%	282	-3%	2%	4%	6%	11%	271
		Myopic	-5%	15%	39%	50%	220%	286	-9%	0%	1%	3%	8%	208
Penetrating	3	PIH	-61%	8%	27%	49%	204%	248	-56%	-1%	3%	7%	49%	204
		Myopic	0%	19%	32%	52%	190%	298	-30%	1%	6%	17%	88%	237
	4	PIH	-28%	18%	37%	55%	232%	289	-19%	2%	4%	7%	48%	245
		Myopic	-2%	21%	35%	53%	219%	299	-28%	1%	4%	7%	61%	236
	5	PIH	-7%	19%	38%	51%	219%	287	-14%	1%	4%	7%	28%	239
		Myopic	-7%	19%	35%	50%	202%	294	-11%	0%	3%	5%	42%	229

Table EC.4 Simulation results for modified heuristics when information updates occur with rate 2 per hour.

Trauma Type	No. of Locations	Policy	vs. Baseline Static Min.	Q1	Med.	Q3	Max	# Sig	vs. Baseline Dynamic Min.	Q1	Med.	Q3	Max.	# Sig
Blunt	3	PIH	-6%	37%	47%	69%	231%	296	-14%	2%	5%	8%	21%	260
		Myopic	-1%	27%	39%	61%	210%	298	-41%	-3%	0%	2%	11%	139
	4	PIH	-1%	37%	47%	61%	241%	296	-8%	3%	5%	7%	18%	278
		Myopic	0%	28%	40%	54%	226%	298	-26%	-2%	1%	2%	10%	167
	5	PIH	-3%	14%	44%	56%	241%	281	-4%	2%	4%	6%	13%	264
		Myopic	-4%	14%	38%	48%	214%	284	-11%	-1%	1%	2%	8%	165
Penetrating	3	PIH	-61%	10%	28%	51%	204%	261	-55%	0%	4%	7%	33%	215
		Myopic	-1%	17%	30%	51%	186%	298	-33%	1%	6%	13%	57%	226
	4	PIH	-23%	20%	37%	56%	232%	293	-14%	2%	5%	8%	40%	252
		Myopic	0%	19%	34%	52%	216%	299	-26%	0%	4%	7%	47%	225
	5	PIH	-6%	19%	38%	51%	219%	289	-12%	1%	5%	7%	22%	243
		Myopic	-6%	18%	33%	49%	195%	292	-8%	0%	3%	5%	30%	205

EC.5. Additional Figures

Figure EC.1 Boxplots for improvement over BLS (nearest facility) policy by number of facilities used under the nearest facility policy (blunt injury).

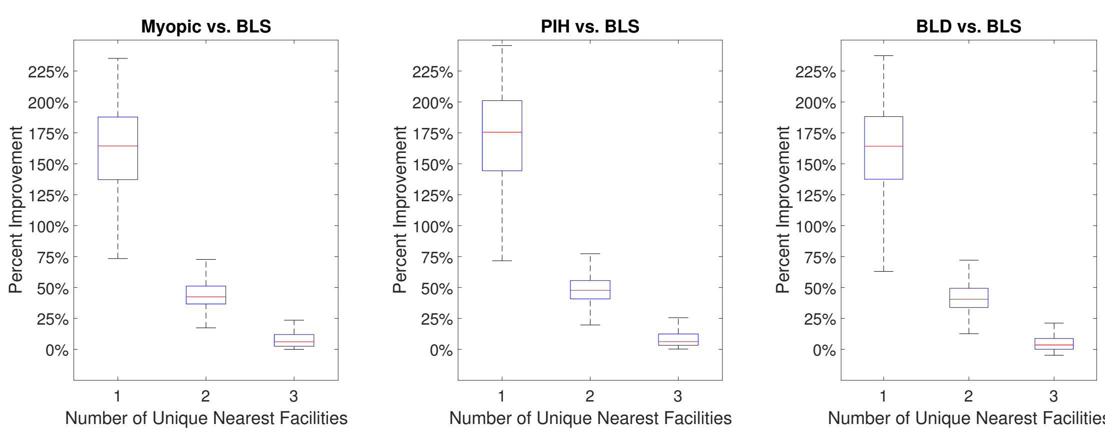


Figure EC.2 Improvement over BLS (nearest facility) policy by distance to farthest facility (blunt injury). Fitted line slope is statistically significant at the 0.05 level for all cases with 1 or 2 unique nearest facilities.

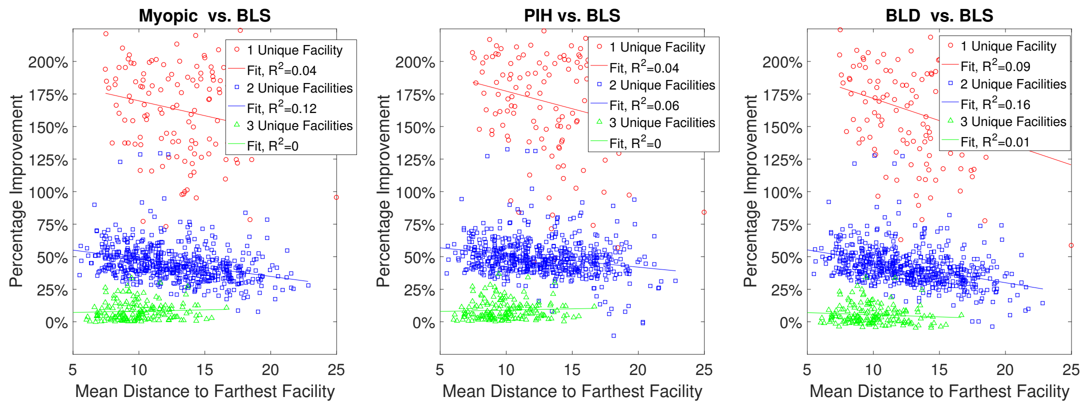


Figure EC.3 Improvement of PIH over Myopic by distance to nearest facility (blunt injury). Fitted line slope is statistically significant at the 0.05 level for all cases except 5 locations.

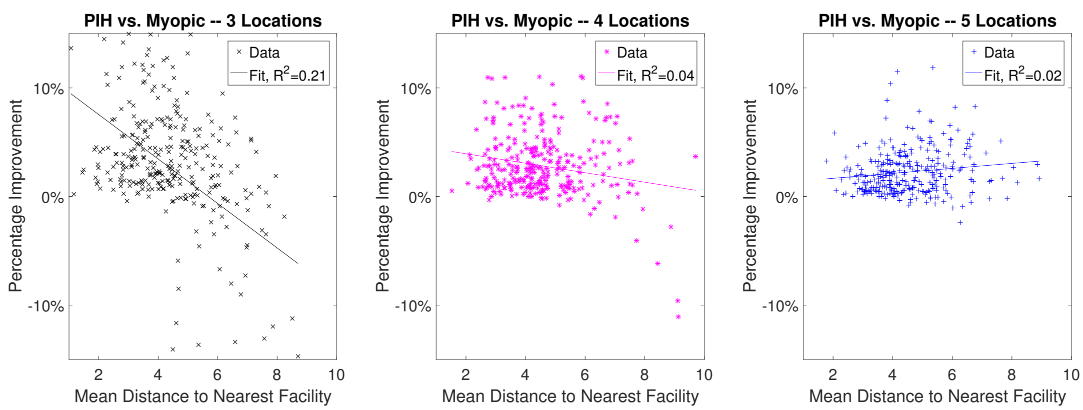


Figure EC.4 Improvement of PIH over BLD by distance to nearest facility (blunt injury). Fitted line slope is statistically significant at the 0.05 level for all cases except 5 locations.

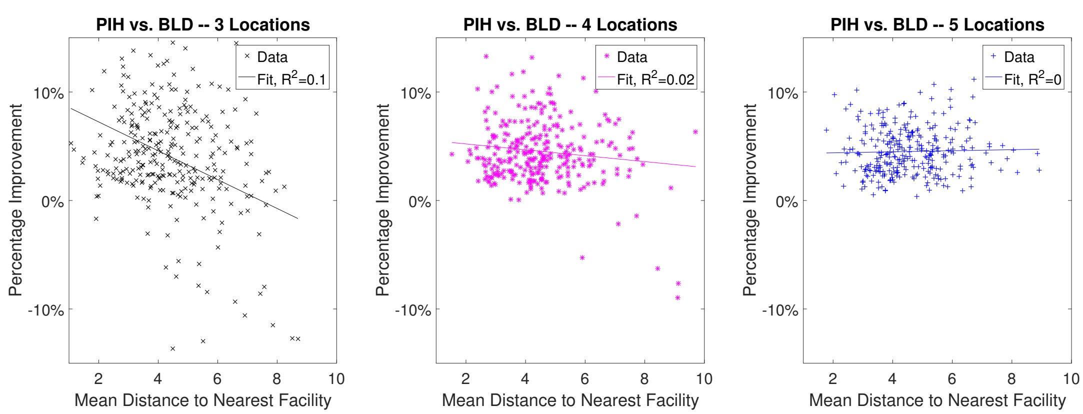
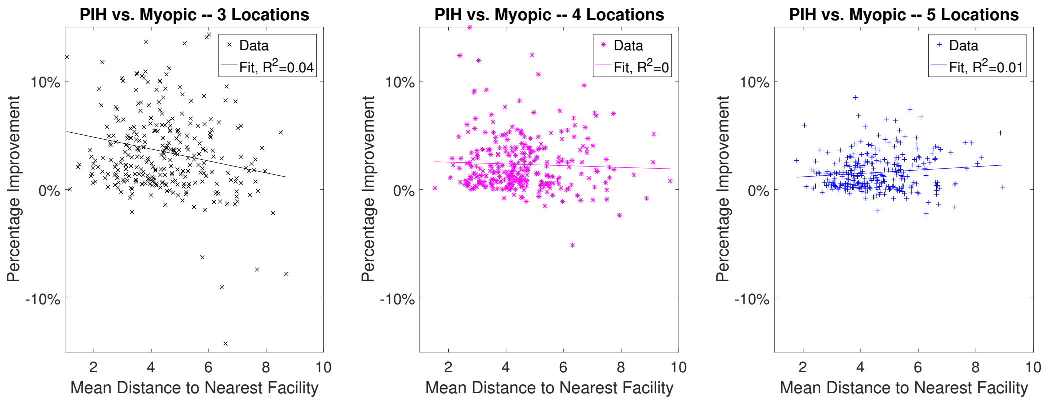


Figure EC.5 Improvement of PIH over Myopic by distance to nearest facility with double the number of ambulances (penetrating injury). Fitted line slope is statistically significant at the 0.05 level only for the graph with 3 locations.



EC.6. Additional Tables

Table EC.5 Parameters used in the randomized simulation study of Section 5.

Medical Facilities	
Number of facilities	3
Geographic x coordinate (in miles)	Uniform $[0, 10]$
Geographic y coordinate for 1st & 2nd facilities (in miles)	Uniform $[0, 10]$
Geographic y coordinate for 3rd facility (in miles)	Uniform $[25, 40]$
Probability of Level I TC	0.58
Number of beds (Level I TC)	Uniform $\{34, 35, \ldots, 44\}$
Number of beds (Level II TC)	Uniform $\{20, 21, \ldots, 30\}$
Service time (Level I TC, Blunt)	Lognormal with log mean 5.39 min. & log s.d. 0.89 min.
Service time (Level II TC, Blunt)	Lognormal with log mean 5.28 min. & log s.d. 0.69 min.
Service time (Level I TC, Penetrating)	Exponential with mean 204 min.
Service time (Level II TC, Penetrating)	Exponential with mean 168 min.
Percentage of beds available initially	10%
Rewards $r_j, \forall j \in \mathcal{F}$	1
Casualty Locations	
Number of locations	$3, 4$, or 5
Geographic x coordinate (in miles)	Uniform $[0, 10]$
Geographic y coordinate (in miles)	Uniform $[0, 10]$
Number of casualties per location	Uniform $\{5, 6, \ldots, 75\}$
Number of dedicated ambulances per location	1
Other	
Number of flexible ambulances	2
Hours taken to travel d units of distance	Lognormal with mean $0.025d$ and s.d. $0.01d$
Distance between (x_1, y_1) and (x_2, y_2)	$\lvert x_1 - x_2\rvert + \lvert y_1 - y_2\rvert$
Discount rate α (Blunt)	0.270
Discount rate α (Penetrating)	0.408

Table EC.6 Notation.

Symbol	Meaning
$a^{\cdot}(\cdot)$	marginal expected future reward due to arrivals
b_j	number of beds at facility $j \in \mathcal{F}$
$c^{\cdot}(\cdot)$	marginal expected future reward due to service completions
$\mathbf{d} = [d_{ij}]$	decision variable: number of dedicated ambulances to send from $i \in \mathcal{L}$ to $j \in \mathcal{F}$
$\mathbf{e}_j$	unit-norm vector having one in component j and zero in all other components
$\mathbf{f} = [f_{ij}]$	decision variable: number of flexible ambulances to send from $i \in \mathcal{L}$ to $j \in \mathcal{F}$
$m_{ij}^{\cdot}$	marginal expected discounted reward earned from going to facility $j \in \mathcal{F}$ from location $i \in \mathcal{L}$
n_i	number of dedicated ambulances at location $i \in \mathcal{L}$
n^f	number of flexible ambulances
r_j	reward due to service completion at facility $j \in \mathcal{F}$
t	epoch in finite horizon model
$\mathbf{w} = [w_i]$	number of casualties at location $i \in \mathcal{L}$
$\mathbf{x} = [x_j]$	number of casualties waiting at facility $j \in \mathcal{F}$
α	discount factor
β	uniformization constant
λ_j	total arrival rate to station $j \in \mathcal{F}$ under a Bernoulli-splitting policy
μ_j	service rate at facility $j \in \mathcal{F}$
ρ_{ij}	Bernoulli-splitting probability of assigning a dedicated ambulance to the (i, j) route
τ_{ij}	transportation rate from location $i \in \mathcal{L}$ to facility $j \in \mathcal{F}$
θ_{ij}	Bernoulli-splitting probability of assigning a flexible ambulance to the (i, j) route
T_{ij}	operator interchanging the ith and jth component of a vector
$V^{\cdot}(\cdot)$	value function
Γ	set of Bernoulli-splitting policies
Λ_j	random variable denoting belief about the system state at facility $j \in \mathcal{F}$
$\mathcal{A}(\cdot)$	action set in the MDP
$\mathcal{F}$	facility set
$\mathcal{L}$	location set
$\mathcal{S}$	state space

Table EC.7 Simulation results with double the number of ambulances.

Trauma Type	No. of Locations	Policy	vs. Baseline Static						vs. Baseline Dynamic					
			Min.	Q1	Med.	Q3	Max	# Sig	Min.	Q1	Med.	Q3	Max.	# Sig
Blunt	3	PIH	0%	42%	51%	74%	256%	299	-2%	2%	4%	6%	19%	296
		Myopic	0%	38%	48%	67%	250%	299	-9%	1%	2%	3%	15%	243
	4	PIH	0%	40%	50%	61%	254%	299	0%	3%	4%	6%	11%	300
		Myopic	0%	38%	48%	58%	249%	300	-1%	2%	3%	4%	8%	285
	5	PIH	-1%	13%	46%	56%	253%	299	0%	3%	4%	6%	10%	300
		Myopic	-1%	13%	44%	54%	244%	295	-3%	2%	3%	5%	10%	293
Penetrating	3	PIH	0%	39%	48%	70%	243%	300	-14%	4%	6%	9%	52%	286
		Myopic	0%	33%	42%	64%	233%	300	-14%	1%	3%	5%	53%	237
	4	PIH	1%	39%	47%	61%	241%	300	0%	4%	6%	9%	20%	300
		Myopic	1%	35%	44%	58%	234%	300	-7%	2%	4%	6%	14%	275
	5	PIH	-2%	16%	45%	54%	234%	299	0%	4%	6%	9%	17%	299
		Myopic	0%	15%	42%	51%	224%	298	-6%	3%	5%	7%	16%	291

Table EC.8 Simulation results with 50% reduction in travel time coefficient of variation.

Trauma Type	No. of Locations	Policy	vs. Baseline Static Min.	Q1	Med.	Q3	Max	# Sig	vs. Baseline Dynamic Min.	Q1	Med.	Q3	Max.	# Sig
Blunt	3	PIH	-3%	37%	48%	68%	246%	299	-19%	2%	4%	8%	33%	262
		Myopic	0%	32%	41%	66%	231%	300	-12%	-1%	1%	4%	25%	198
	4	PIH	1%	38%	48%	63%	241%	300	-7%	3%	4%	6%	21%	293
		Myopic	1%	33%	43%	59%	235%	300	-9%	1%	2%	3%	12%	237
	5	PIH	0%	17%	46%	56%	245%	300	0%	3%	4%	6%	12%	299
		Myopic	0%	16%	41%	52%	226%	299	-11%	1%	2%	4%	9%	267
Penetrating	3	PIH	-60%	8%	30%	52%	202%	253	-54%	2%	7%	16%	81%	237
		Myopic	0%	19%	35%	55%	192%	300	-15%	4%	12%	27%	134%	262
	4	PIH	-31%	21%	39%	55%	223%	290	-15%	4%	7%	13%	65%	267
		Myopic	2%	23%	36%	55%	213%	300	-7%	3%	7%	14%	90%	270
	5	PIH	-8%	19%	40%	52%	215%	298	-10%	4%	7%	12%	40%	285
		Myopic	1%	19%	35%	49%	200%	300	-7%	3%	5%	11%	52%	280

Table EC.9 Simulation results with a small number of patients (5–15 per location).

Trauma Type	No. of Locations	Policy	vs. Baseline Static Min.	Q1	Med.	Q3	Max	# Sig	vs. Baseline Dynamic Min.	Q1	Med.	Q3	Max.	# Sig
Blunt	3	PIH	-13%	14%	21%	34%	134%	294	-11%	2%	4%	8%	28%	267
		Myopic	-2%	5%	13%	28%	115%	290	-21%	-5%	-1%	2%	21%	111
	4	PIH	-1%	20%	28%	43%	179%	299	-6%	3%	4%	6%	17%	291
		Myopic	1%	9%	20%	35%	160%	300	-14%	-5%	-1%	2%	18%	111
	5	PIH	0%	18%	32%	48%	168%	300	-1%	3%	4%	6%	10%	297
		Myopic	0%	12%	24%	41%	152%	299	-15%	-3%	0%	2%	9%	148
Penetrating	3	PIH	-48%	-1%	8%	21%	105%	214	-36%	3%	7%	15%	71%	252
		Myopic	-2%	3%	8%	19%	92%	290	-15%	1%	8%	21%	93%	237
	4	PIH	-24%	6%	17%	31%	156%	271	-13%	3%	7%	12%	42%	273
		Myopic	0%	5%	13%	27%	133%	299	-10%	0%	5%	11%	80%	219
	5	PIH	-9%	11%	21%	38%	151%	293	-8%	4%	6%	11%	33%	286
		Myopic	0%	8%	17%	31%	131%	298	-10%	0%	4%	8%	30%	223

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