E-Companion to the Paper "When to Triage in Service Systems with Hidden Customer Class Identities?"

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Proof of Lemma 3. We first prove part (i), i.e., f1) and f2) are preserved under operator L. For any $\boldsymbol{x} \in \mathcal{S}$ and $x_0 \geq 1, x_1 = 0$, by Theorem 3 (i)&(ii) and f2), we have

$$\begin{aligned}
&q_1 u^{-1}[\tau_1^{-1}(Lv(\boldsymbol{x}+\boldsymbol{e_2}) - Lv(\boldsymbol{x})) - \tau_0^{-1}(Lv(\boldsymbol{x}+\boldsymbol{e_2}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))] \\
&= \lambda q_1 u^{-1}[\tau_1^{-1}(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}+\boldsymbol{e_1})) - \tau_0^{-1}(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}+\boldsymbol{e_2}))] + q_1 u^{-1}(r_1/\tau_1 - r_0/\tau_0) \\
&\quad + (u^{-1}+\tau_0^{-1}+\tau_2^{-1}) q_1 u^{-1}[\tau_1^{-1}(v(\boldsymbol{x}+\boldsymbol{e_2}) - v(\boldsymbol{x})) - \tau_0^{-1}(v(\boldsymbol{x}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))] \\
&\leq (1-\alpha-\tau_1^{-1}) r_0 \tau_0^{-1} + q_1 u^{-1}(r_1/\tau_1 - r_0/\tau_0) = r_0\tau_0^{-1} + (1+\alpha\tau_1)(\tilde{u}(\alpha) - u) r_0/(\tau_0\tau_1 u) \leq r_0 \tau_0^{-1},
\end{aligned}$$

where the first inequality holds because of f1) and the last inequality holds because of the assumption $u \geq \tilde{u}(\alpha)$, which completes the proof that f1) is preserved under L.

Next, we show that f2) is preserved under operator L. For $\boldsymbol{x} = (x_0, 0, x_2)$ with $x_0 \geq 2$ and $x_2 \geq 0$, by Theorem 3 (i)&(ii) and f2), we have

$$\begin{aligned}
LG(\boldsymbol{x}) &= \lambda G(\boldsymbol{x}+\boldsymbol{e_1}) + (u^{-1}+\tau_1^{-1}+\tau_2^{-1}) G(\boldsymbol{x}) + \tau_0^{-1} G(\boldsymbol{x}-\boldsymbol{e_1}) \\
&\quad + r_0\tau_0^{-1} - q_1 u^{-1}[(\tau_1^{-1}-\tau_0^{-1})[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1})] + \tau_0^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1})]] \\
&\geq r_0\tau_0^{-1} - q_1 u^{-1}[\tau_1^{-1}(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1})) - \tau_0^{-1}(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}))] \geq 0,
\end{aligned}$$

where the first inequality holds because of f2) and Theorem 3 (iii), and the second inequality holds because of f1), which completes the proof that f2) is preserved under L.

We prove part (ii) by verifying the conditions in Theorem 6.11.3 of Puterman (2005). It is obvious that the state space $\mathcal{S}$ is countable and we have already verified Assumptions 6.10.1 and 6.10.2 of Puterman (2005) in our proof of Proposition 4. Hence, we only need to show that conditions (a), (b), and (c) in Theorem 6.11.3 of Puterman (2005) hold. Condition (a) holds by part (i) of Lemma 3. Next, consider a stationary policy π that never triages class 0 customers and serves customers in the following priority order: class 1, class 0, and class 2. From the optimality equations, we find that $v \in \mathbb{F}$ implies that policy π is an optimal policy. Hence, (b) holds. Finally, condition (c) holds, i.e., $\mathbb{F}$ is closed, because the limit of any convergent sequence of functions that satisfy f1) and f2), will satisfy them as well, which concludes the proof. □

We need the following lemma to prove Lemma 4.

LEMMA EC.1. *Suppose that* (14) *holds,* $r_1 \geq r_0 \geq r_2$, $\tau_0 = \tau_1 = \tau_2 = \tau$, $u < \tilde{u}(\alpha)$, *and* $0 < \alpha < u^{-1} - \lambda$. *Then* β_1 *and* β_2, *as defined in* (18) *and* (19), *satisfy the following conditions:*

$$\frac{r_2}{\alpha}(1-\beta_1) \leq (q_2 u^{-1} - \tau^{-1})(r_0 - r_2)\tau^2, \tag{EC.1}$$

$$0 \leq \beta_2 < \beta_1 < 1, \tag{EC.2}$$

$$\lambda\beta_1(1-\beta_2) - \tau^{-1}(1-\beta_1) + \alpha\beta_1 = 0, \tag{EC.3}$$

$$\lambda\beta_2(1-\beta_2) - \tau^{-1}(1-\beta_2) + \alpha\beta_2 \leq 0, \tag{EC.4}$$

$$\lambda\beta_2(1-\beta_2) - u^{-1}(\beta_1-\beta_2) + \alpha\beta_2 \leq 0. \tag{EC.5}$$

Proof: We can rewrite (EC.1) as $\beta_1 \geq 1 - \frac{\alpha\tau^2}{r_2}(q_2 u^{-1} - \tau^{-1})(r_0 - r_2)$, whose right-hand-side is denoted by f. By (14) and the assumption that $u < \tilde{u}(\alpha)$, we know $\lambda(1 + u/\tau) < \tau^{-1}$. Then, it is easy to show that $\beta_1 = f = 1$ when $\alpha = 0$, and

$$\frac{d\beta_1}{d\alpha} = \frac{1}{2\lambda(1+u\tau^{-1})}\left[1 - \frac{\lambda(1+u\tau^{-1}) + \tau^{-1} + \alpha}{\sqrt{(\lambda(1+u\tau^{-1}) + \tau^{-1} + \alpha)^2 - 4\lambda(1+u\tau^{-1})\tau^{-1}}}\right] < 0,$$

$$\left.\frac{d\beta_1}{d\alpha}\right|_{\alpha=0} = -\frac{1}{\tau^{-1} - \lambda(1+u\tau^{-1})}, \text{ and } \frac{df}{d\alpha} = -\frac{\tau^2}{r_2}\left(q_2 u^{-1} - \tau^{-1}\right)(r_0 - r_2).$$

Note that $\left.\frac{d\beta_1}{d\alpha}\right|_{\alpha=0} \geq \left.\frac{df}{d\alpha}\right|_{\alpha=0}$ if and only if (7) holds. Now, we can conclude that $\frac{d\beta_1}{d\alpha} \geq \frac{df}{d\alpha}$ for all $\alpha \geq 0$ because

$$\frac{d^2\beta_1}{d\alpha^2} = \frac{2}{\tau}\left[(\lambda(1+u\tau^{-1}) + \tau^{-1} + \alpha)^2 - 4\lambda(1+u\tau^{-1})\tau^{-1}\right]^{-3/2} > \frac{d^2 f}{d\alpha^2} = 0.$$

Hence, $\beta_1 \geq f$ for $\alpha \geq 0$, i.e., (EC.1) holds.

We next prove (EC.2). Since $\beta_1 = 1$ when $\alpha = 0$ and $d\beta_1/d\alpha < 0$, it is obvious that $\beta_1 < 1$ for any $\alpha > 0$. Furthermore, by the definition of β_2, we have $\beta_1 - \beta_2 = (1-\beta_1)u\tau^{-1} > 0$. To prove $\beta_2 \geq 0$, we need to show that $\beta_1 \geq u\tau^{-1}/(1+u\tau^{-1})$, which is equivalent to $(\lambda + \alpha)u \leq 1$ after some algebraic manipulations. Since $\alpha < u^{-1} - \lambda$, we get $\beta_2 \geq 0$, which completes the proof of (EC.2).

To show (EC.3), we plug the expression of β_2 in (19) into (EC.3) and get $-\lambda(1+u\tau^{-1})\beta_1^2 + [\lambda(1+u\tau^{-1}) + \tau^{-1} + \alpha]\beta_1 - \tau^{-1} = 0$. It is straightforward to verify that β_1 defined in (18) is a solution to the above equation. Hence, (EC.3) holds.

Finally, taking the differences of the left-hand sides of (EC.3) and (EC.4), we get

$$\begin{aligned}&[\lambda\beta_1(1-\beta_2) - \tau^{-1}(1-\beta_1) + \alpha\beta_1] - [\lambda\beta_2(1-\beta_2) - \tau^{-1}(1-\beta_2) + \alpha\beta_2] \\ &= \lambda(1-\beta_2)(\beta_1-\beta_2) + \tau^{-1}(\beta_1-\beta_2) + \alpha(\beta_1-\beta_2) \geq 0,\end{aligned}$$

which proves (EC.4). Similarly, taking the differences of the left-hand sides of (EC.3) and (EC.5), we get

$$\begin{aligned}&[\lambda\beta_1(1-\beta_2)-\tau^{-1}(1-\beta_1)+\alpha\beta_1]-[\lambda\beta_2(1-\beta_2)-u^{-1}(\beta_1-\beta_2)+\alpha\beta_2]\\&=\lambda(1-\beta_2)(\beta_1-\beta_2)+\alpha(\beta_1-\beta_2)\geq 0,\end{aligned}$$

which proves (EC.5). □

Proof of Lemma 4. We first show part (i), i.e., h1) through h11) are preserved under L. We first show that L preserves h1) by considering three separate cases. For $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0\geq 2$ and $x_2\geq 0$, we have

$$\begin{aligned}LG(\boldsymbol{x})=&\lambda G(\boldsymbol{x}+\boldsymbol{e_1})+u^{-1}\max\{G(\boldsymbol{x}),0\}+\tau^{-1}\min\{G(\boldsymbol{x}),0\}+q_2u^{-1}\min\{G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}),0\}\\&+\tau^{-1}\max\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\}+q_1(\tau u)^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2})]+r_0\tau^{-1}.\end{aligned}\tag{EC.6}$$

Then, by h1) and h2), each term of $LG(\boldsymbol{x})$ is less than or equal to the corresponding term of $LG(\boldsymbol{x}+\boldsymbol{e_3})$ thus $LG(\boldsymbol{x})\leq LG(\boldsymbol{x}+\boldsymbol{e_3})$. When $\boldsymbol{x}=(1,0,x_2)$ with $x_2\geq 1$, we have

$$LG(\boldsymbol{x})=\lambda G(\boldsymbol{x}+\boldsymbol{e_1})+u^{-1}G(\boldsymbol{x})+\tau^{-2}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]+r_0\tau^{-1}.\tag{EC.7}$$

Then, by h1) and h3), $LG(\boldsymbol{x})\leq LG(\boldsymbol{x}+\boldsymbol{e_3})$ for $\boldsymbol{x}=(1,0,x_2)$ with $x_2\geq 1$. Finally, we have

$$LG(\boldsymbol{e_1})=\lambda G(2\boldsymbol{e_1})+u^{-1}G(\boldsymbol{e_1})+r_0\tau^{-1}.\tag{EC.8}$$

Then, by h1), $LG(\boldsymbol{e_1})\leq LG(\boldsymbol{e_1}+\boldsymbol{e_3})$, which completes the proof that h1) is preserved under L.

Now we show that h2) is preserved under L. For $x_0\geq 0,\ x_1=0$, and $x_2\geq 0$, we have

$$\begin{aligned}&Lv(\boldsymbol{x}+\boldsymbol{e_1})-Lv(\boldsymbol{x}+\boldsymbol{e_2})\\&=\lambda[v(\boldsymbol{x}+2\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})]+\min\{G(\boldsymbol{x}+\boldsymbol{e_1}),0\}+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_2})]+r_0-r_1.\end{aligned}$$

By h1) and h2), we have $Lv(\boldsymbol{x}+\boldsymbol{e_1})-Lv(\boldsymbol{x}+\boldsymbol{e_2})\leq Lv(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_2}+\boldsymbol{e_3})$, which completes the proof that h2) is preserved under L.

Next, we show that h3) is preserved by considering three separate cases. For $\boldsymbol{x}=\boldsymbol{0}$, we have

$$\begin{aligned}&Lv(\boldsymbol{e_3})-Lv(\boldsymbol{0})=\lambda[v(\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{e_1})]+u^{-1}[v(\boldsymbol{e_3})-v(\boldsymbol{0})]+r_2,\\&Lv(2\boldsymbol{e_3})-Lv(\boldsymbol{e_3})=\lambda[v(\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{e_1}+\boldsymbol{e_3})]+\tau^{-1}[v(\boldsymbol{e_3})-v(\boldsymbol{0})]+u^{-1}[v(2\boldsymbol{e_3})-v(\boldsymbol{e_3})]+r_2.\end{aligned}$$

Then, by h3), we obtain $Lv(\boldsymbol{e_3})-Lv(\boldsymbol{0})\leq Lv(2\boldsymbol{e_3})-Lv(\boldsymbol{e_3})$. For $\boldsymbol{x}=(0,0,x_2)$ with $x_2\geq 1$, we have

$$Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})=\lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+\tau^{-1}[v(\boldsymbol{x})-v(\boldsymbol{x}-\boldsymbol{e_3})]+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]+r_2.$$

Then, by h3), we have $Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})\leq Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})$. For $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0\geq 1$ and $x_2\geq 0$, we have

$$Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})=\lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1})]$$
$$+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]+\min\{G(\boldsymbol{x}+\boldsymbol{e_3}),0\}-\min\{G(\boldsymbol{x}),0\}+r_2.$$

To compare $Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})$ and $Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})$ for $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0\geq 1$ and $x_2\geq 0$, we look at three separate cases: (i) If $G(\boldsymbol{x})\geq 0$, then by h1), $G(\boldsymbol{x}+2\boldsymbol{e_3})\geq G(\boldsymbol{x}+\boldsymbol{e_3})\geq 0$, and hence $Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})\geq Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})$ by h3); (ii) If $G(\boldsymbol{x})<0$ and $G(\boldsymbol{x}+2\boldsymbol{e_3})\geq 0$, we have

$$\begin{aligned}
&Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})\\
&\leq \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+u^{-1}[q_1v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})+q_2v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})]+\tau^{-1}v(\boldsymbol{x}+\boldsymbol{e_3})+r_2\\
&\quad -u^{-1}[q_1v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})+q_2v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]-\tau^{-1}v(\boldsymbol{x})\\
&=\lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+q_1u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})]+\tau^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]\\
&\quad +q_2u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]+r_2, \text{ and}\\
&Lv(\boldsymbol{x}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}+\boldsymbol{e_3})\\
&\geq \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})]+\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]+u^{-1}[v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3})]+r_2,
\end{aligned}$$

which leads to

$$\begin{aligned}
&Lv(\boldsymbol{x}+2\boldsymbol{e_3})-2Lv(\boldsymbol{x}+\boldsymbol{e_3})+Lv(\boldsymbol{x})\\
&\geq \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))]\\
&\quad +\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]+u^{-1}[v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3})]-\tau^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]\\
&\quad -q_1u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})]-q_2u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\
&\geq q_1u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))]\\
&\quad +q_1u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))]\\
&\quad +q_2u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\quad -\tau^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\geq (q_2u^{-1}-\tau^{-1})[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]>0,
\end{aligned}$$

where the second inequality holds by h3), the third inequality holds by h2) and h3), and the last inequality holds by h4) and the condition $u<\tilde{u}(\alpha)$ and the fact $\tilde{u}(\alpha)<q_2\tau$. (iii) If $G(\boldsymbol{x}+2\boldsymbol{e_3})<0$, then by h1), we have $G(\boldsymbol{x})\leq G(\boldsymbol{x}+\boldsymbol{e_1})<0$. Then, we obtain

$$\begin{aligned}
&Lv(\boldsymbol{x}+2\boldsymbol{e_3})-2Lv(\boldsymbol{x}+\boldsymbol{e_3})+Lv(\boldsymbol{x})\\
&=\lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))]\\
&\quad +q_1u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))]\\
&\quad +q_2u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+3\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&\quad +\tau^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))-(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))]\geq 0,
\end{aligned}$$

where the inequality holds by h3). This completes the proof that h3) is preserved under L.

We next show that h4) is preserved by considering two separate cases. For $\boldsymbol{x} = (1, 0, x_2)$ with $x_2 \geq 0$, by h1) and h4), we have

$$\begin{aligned}&[Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})] - [Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\ &= \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})) - (v(\boldsymbol{x}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_3}))]\\ &\quad + u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})) - (v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))] \geq 0.\end{aligned}$$

For $\boldsymbol{x} = (x_0, 0, x_2)$ with $x_0 \geq 2$ and $x_2 \geq 0$, we have

$$\begin{aligned}&[Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})] - [Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\ &= \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})) - (v(\boldsymbol{x}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_3}))] + \tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}))\\ &\quad - (v(\boldsymbol{x}-2\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_3}))] + u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})) - (v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\ &\quad + \min\{G(\boldsymbol{x}+\boldsymbol{e_3}), 0\} + \min\{G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}), 0\} - \min\{G(\boldsymbol{x}), 0\} - \min\{G(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}), 0\}. \qquad \text{(EC.9)}\end{aligned}$$

To compare $Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})$ and $Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})$, we look at three separate cases. (i) When $G(\boldsymbol{x}+\boldsymbol{e_3}) \geq 0$ and $G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) \geq 0$, by h1) we have $G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{2e_3}) \geq 0$. Hence, by (EC.9)

$$\begin{aligned}&[Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})] - [Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\ &\geq \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})] - (v(\boldsymbol{x}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_3}))] + \tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1})) - (v(\boldsymbol{x}-2\boldsymbol{e_1}\\ &\quad +2\boldsymbol{e_3}) - v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_3}))] + u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})] - (v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))] \geq 0,\end{aligned}$$

where the last inequality holds because of h4).

(ii) When $G(\boldsymbol{x}+\boldsymbol{e_3}) < 0$ and $G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) \geq 0$, by h1) we have $G(\boldsymbol{x}) < 0$ and $G(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) > 0$. Hence, by (EC.9), we have

$$\begin{aligned}&[Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})] - [Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\ &= \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})) - (v(\boldsymbol{x}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_3}))]\\ &\quad + \tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1})) - (v(\boldsymbol{x}-2\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_3}))]\\ &\quad + u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})) - (v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))] + G(\boldsymbol{x}+\boldsymbol{e_3}) - G(\boldsymbol{x}) \geq 0,\end{aligned}$$

where the last inequality holds because of h1) and h4).

(iii) Otherwise, i.e., $G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) < 0$, we have $G(\boldsymbol{x}+\boldsymbol{e_3}) < 0$ by h5). Hence, by (EC.9), we have

$$
\begin{aligned}
&[Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})]-[Lv(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-Lv(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})]\\
\geq{}& \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))-(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))]+\tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}))\\
&-(v(\boldsymbol{x}-2\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_3}))]+u^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&+G(\boldsymbol{x}+\boldsymbol{e_3})+G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-G(\boldsymbol{x})-G(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})\\
={}& \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1}))-(v(\boldsymbol{x}+2\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_3}))]\\
&+\tau^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x}))-(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))]\\
&+q_1u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))-(v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}+2\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}))]\\
&+q_2u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))-(v(\boldsymbol{x}-2\boldsymbol{e_1}+3\boldsymbol{e_3})-v(\boldsymbol{x}-2\boldsymbol{e_1}+2\boldsymbol{e_3}))]\geq 0,
\end{aligned}
$$

where the last inequality follows from h4). This completes the proof that h4) is preserved under L.

We next show that h5) is preserved by considering three separate cases. When $x_0 \geq 2,\ x_1 = 0$, and $x_2 \geq 0$, then $LG(\boldsymbol{x})$ satisfies (EC.6), which also gives

$$
LG(\boldsymbol{x}+\boldsymbol{e_1})=\lambda G(\boldsymbol{x}+2\boldsymbol{e_1})+u^{-1}G(\boldsymbol{x}+\boldsymbol{e_1})+r_0\tau^{-1}+\tau^{-1}G(\boldsymbol{x})+q_1(\tau u)^{-1}[v(\boldsymbol{x})-v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})]. \quad \text{(EC.10)}
$$

Then, by h5) and h6), $LG(\boldsymbol{x}) \geq LG(\boldsymbol{x}+\boldsymbol{e_1})$. When $\boldsymbol{x}=(1,0,x_2)$ with $x_2 \geq 1$, then $LG(\boldsymbol{x})$ satisfies (EC.7) and $LG(\boldsymbol{x}+\boldsymbol{e_1})$ is given by (EC.10). By h5), h7) and h8) we have

$$
LG(\boldsymbol{x}+\boldsymbol{e_1})-LG(\boldsymbol{x}) \leq \tau^{-1}G(\boldsymbol{x})-q_1(\tau u)^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})-v(\boldsymbol{x})]-\tau^{-2}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})] \leq 0.
$$

When $\boldsymbol{x}=\boldsymbol{e_1}$, $LG(\boldsymbol{e_1})$ is given by (EC.8) and $LG(2\boldsymbol{e_1})$ is given by (EC.6). By h5), h7), and h10), we have

$$
LG(2\boldsymbol{e_1})-LG(\boldsymbol{e_1}) \leq \tau^{-1}G(\boldsymbol{e_1})+q_1(\tau u)^{-1}(v(\boldsymbol{e_1})-v(\boldsymbol{e_2})) \leq \tau^{-1}r_0(1-\tilde{u}(\alpha)/u)<0,
$$

where we use the condition that $u<\tilde{u}(\alpha)$.

Next, we show that h6) is preserved. For any $\boldsymbol{x}\in\mathcal{S}$ with $x_1=0$, we have

$$
\begin{aligned}
Lv(\boldsymbol{x}+\boldsymbol{e_1})-Lv(\boldsymbol{x}+\boldsymbol{e_2}) &= \lambda[v(\boldsymbol{x}+2\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})]+\min\{G(\boldsymbol{x}+\boldsymbol{e_1}),0\}\\
&\quad+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_2})]+r_0-r_1, \text{ and}\\
Lv(\boldsymbol{x}+2\boldsymbol{e_1})-Lv(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2}) &= \lambda[v(\boldsymbol{x}+3\boldsymbol{e_1})-v(\boldsymbol{x}+2\boldsymbol{e_1}+\boldsymbol{e_2})]+\min\{G(\boldsymbol{x}+2\boldsymbol{e_1}),0\}\\
&\quad+u^{-1}[v(\boldsymbol{x}+2\boldsymbol{e_1})-v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})]+r_0-r_1.
\end{aligned}
$$

Then, by h5) and h6), it is obvious that $Lv(\boldsymbol{x}+\boldsymbol{e_1})-Lv(\boldsymbol{x}+\boldsymbol{e_2}) \geq Lv(\boldsymbol{x}+2\boldsymbol{e_1})-Lv(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})$.

Now, we show that h7) is preserved. For $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0 \geq 0$ and $x_2 \geq 0$, we have

$$
\begin{aligned}
q_1[Lv(\boldsymbol{x}+\boldsymbol{e_2})-Lv(\boldsymbol{x}+\boldsymbol{e_1})] &\geq q_1[\lambda(v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_2})-v(\boldsymbol{x}+2\boldsymbol{e_1}))+u^{-1}(v(\boldsymbol{x}+\boldsymbol{e_2})-v(\boldsymbol{x}+\boldsymbol{e_1}))+r_1-r_0]\\
&\geq (\lambda+u^{-1})\tilde{u}(\alpha)r_0+q_1(r_1-r_0)=(\lambda+u^{-1})\tilde{u}(\alpha)r_0+q_2(r_0-r_2)=\tilde{u}(\alpha)r_0,
\end{aligned}
$$

where the inequality holds by h7).

Next, we show that h8) is preserved by considering three separate cases. When $\boldsymbol{x} = \boldsymbol{e_1} + \boldsymbol{e_3}$, by (EC.7) we have

$$
\begin{aligned}
& LG(\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}[Lv(\boldsymbol{e_3}) - Lv(\boldsymbol{0})] \\
\le\ & \lambda[G(2\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}(v(\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{e_1}))] + u^{-1}[G(\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}(v(\boldsymbol{e_3}) - v(\boldsymbol{0}))] + \tau^{-2}[v(\boldsymbol{e_3}) - v(\boldsymbol{0})] \\
& + (r_0 - r_2)\tau^{-1} \\
\le\ & (\lambda + u^{-1})\tilde{u}(\alpha) r_0 u^{-1} + u^{-1}[G(\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}(v(\boldsymbol{e_3}) - v(\boldsymbol{0})) - \tilde{u}(\alpha) r_0 u^{-1}] + q_2 u^{-1}(r_0 - r_2) \\
=\ & \tilde{u}(\alpha) r_0 u^{-1} + u^{-1}[G(\boldsymbol{e_1}+\boldsymbol{e_3}) - \tau^{-1}(v(\boldsymbol{e_3}) - v(\boldsymbol{0})) - \tilde{u}(\alpha) r_0 u^{-1}] \le \tilde{u}(\alpha) r_0 u^{-1},
\end{aligned}
$$

where h8), h11), and (EC.1) are used in the second inequality, and the last inequality follows by h8) if $G(\boldsymbol{e_1}+\boldsymbol{e_3}) > 0$.

When $\boldsymbol{x} = (1, 0, x_2)$ with $x_2 \ge 2$, by (EC.7) we have

$$
\begin{aligned}
& LG(\boldsymbol{x}) - \tau^{-1}[Lv(\boldsymbol{x}-\boldsymbol{e_1}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})] \\
\le\ & \lambda[G(\boldsymbol{x}+\boldsymbol{e_1}) - \tau^{-1}[v(\boldsymbol{x}) - v(\boldsymbol{x}-\boldsymbol{e_3})]] + u^{-1}[G(\boldsymbol{x}) - \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]] \\
& + \tau^{-2}[(v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})) - (v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}-2\boldsymbol{e_3}))] + \tau^{-1}(r_0 - r_2) \\
\le\ & (\lambda + u^{-1})\tilde{u}(\alpha) r_0 u^{-1} + u^{-1}[G(\boldsymbol{x}) - \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})] - \tilde{u}(\alpha) r_0 u^{-1}] + q_2 u^{-1}(r_0 - r_2) \\
=\ & \tilde{u}(\alpha) r_0 u^{-1} + u^{-1}[G(\boldsymbol{x}) - \tau^{-1}(v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})) - \tilde{u}(\alpha) r_0 u^{-1}] \le \tilde{u}(\alpha) r_0 u^{-1},
\end{aligned}
$$

where the first inequality follows by h8) and h9), and the last inequality follows by h8) if $G(\boldsymbol{x}) > 0$. When $x_0 \ge 2,\ x_1 = 0$, and $x_2 \ge 1$, by (EC.6) we have

$$
\begin{aligned}
& LG(\boldsymbol{x}) - \tau^{-1}(Lv(\boldsymbol{x}-\boldsymbol{e_1}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})) \\
=\ & \lambda[G(\boldsymbol{x}+\boldsymbol{e_1}) - \tau^{-1}[v(\boldsymbol{x}) - v(\boldsymbol{x}-\boldsymbol{e_3})]] + u^{-1}[\max\{G(\boldsymbol{x}), 0\} - \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]] \\
& + \tau^{-1}[\max\{G(\boldsymbol{x}-\boldsymbol{e_1}), 0\} - \tau^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}) - v(\boldsymbol{x}-2\boldsymbol{e_1}-\boldsymbol{e_3})]] - q_1(\tau u)^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2}) - v(\boldsymbol{x}-\boldsymbol{e_1})] \\
& + \tau^{-1}(\min\{G(\boldsymbol{x}), 0\} - \min\{G(\boldsymbol{x}-\boldsymbol{e_1}), 0\}) + q_2 u^{-1}\min\{G(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}), 0\} \\
& + \tau^{-1}\min\{G(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3}), 0\} + (r_0 - r_2)\tau^{-1} \\
\le\ & \lambda[G(\boldsymbol{x}+\boldsymbol{e_1}) - \tau^{-1}[v(\boldsymbol{x}) - v(\boldsymbol{x}-\boldsymbol{e_3})]] + u^{-1}[\max\{G(\boldsymbol{x}), 0\} - \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})]] \\
& + \tau^{-1}[\max\{G(\boldsymbol{x}-\boldsymbol{e_1}), 0\} - \tau^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}) - v(\boldsymbol{x}-2\boldsymbol{e_1}-\boldsymbol{e_3})]] - \tilde{u}(\alpha) r_0(\tau u)^{-1} + (r_0 - r_2)\tau^{-1},
\end{aligned}
$$

where the last inequality holds because of h5) and h7). Then, by h8) we have

$$
\begin{aligned}
& LG(\boldsymbol{x}) - \tau^{-1}(Lv(\boldsymbol{x}-\boldsymbol{e_1}) - Lv(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})) \\
\le\ & (\lambda + u^{-1})\tilde{u}(\alpha) r_0 u^{-1} + u^{-1}[\max\{G(\boldsymbol{x}), 0\} - \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})] - \tilde{u}(\alpha) r_0 u^{-1}] \\
& + \tau^{-1}[\max\{G(\boldsymbol{x}-\boldsymbol{e_1}), 0\} - \tau^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}) - v(\boldsymbol{x}-2\boldsymbol{e_1}-\boldsymbol{e_3})] - \tilde{u}(\alpha) r_0 u^{-1}] + (r_0 - r_2)\tau^{-1} \\
=\ & \tilde{u}(\alpha) r_0 u^{-1} + u^{-1}[\max\{G(\boldsymbol{x}), 0\} - \tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}) - v(\boldsymbol{x}-\boldsymbol{e_1}-\boldsymbol{e_3})] - \tilde{u}(\alpha) r_0 u^{-1}] + (r_0 - r_2)\tau^{-1} \\
& + \tau^{-1}[\max\{G(\boldsymbol{x}-\boldsymbol{e_1}), 0\} - \tau^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}) - v(\boldsymbol{x}-2\boldsymbol{e_1}-\boldsymbol{e_3})] - \tilde{u}(\alpha) r_0 u^{-1}] - (\alpha + \tau^{-1})\tilde{u}(\alpha) r_0 u^{-1} \\
\le\ & \tilde{u}(\alpha) r_0 u^{-1} + (r_0 - r_2)(\tau^{-1} - q_2 u^{-1}) < \tilde{u}(\alpha) r_0 u^{-1},
\end{aligned}
$$

where the last inequality holds because $r_0 > r_2$ and $u < \tilde{u}(\alpha) < q_2\tau$.

Now, we show that h9) is preserved under L by considering four separate cases. When $\boldsymbol{x} \in \mathcal{S}$ and $x_1 \geq 1$, by h9) we have

$$\begin{aligned}
&[Lv(\boldsymbol{x}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}+\boldsymbol{e_3})] - [Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})] \\
&= \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})) - (v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1}))] + \tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_2}+2\boldsymbol{e_3}) - v(\boldsymbol{x} \\
&\quad -\boldsymbol{e_2}+\boldsymbol{e_3})) - (v(\boldsymbol{x}-\boldsymbol{e_2}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_2}))] + u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_3})) - (v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x}))] \\
&\leq (1-\alpha)(q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2) < (q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2).
\end{aligned}$$

When $\boldsymbol{x} = \boldsymbol{0}$, by h9), h11), and (EC.1), we have

$$\begin{aligned}
&[Lv(2\boldsymbol{e_3}) - Lv(\boldsymbol{e_3})] - [Lv(\boldsymbol{e_3}) - Lv(\boldsymbol{0})] \\
&= \lambda[(v(\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{e_1}+\boldsymbol{e_3})) - (v(\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{e_1}))] + \tau^{-1}(v(\boldsymbol{e_3}) - v(\boldsymbol{0})) \\
&\quad + u^{-1}[(v(2\boldsymbol{e_3}) - v(\boldsymbol{e_3})) - (v(\boldsymbol{e_3}) - v(\boldsymbol{0}))] \leq (1-\alpha)(q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2) < (q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2).
\end{aligned}$$

When $\boldsymbol{x} = (0, 0, x_2)$ with $x_2 \geq 1$, by h9) we have

$$\begin{aligned}
&[Lv(\boldsymbol{x}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}+\boldsymbol{e_3})] - [Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})] \\
&= \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})) - (v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1}))] + \tau^{-1}[(v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})) \\
&\quad - (v(\boldsymbol{x}) - v(\boldsymbol{x}-\boldsymbol{e_3}))] + u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_3})) - (v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x}))] \\
&\leq (1-\alpha)(q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2) < (q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2).
\end{aligned}$$

Finally, when $\boldsymbol{x} = (x_0, 0, x_2)$ with $x_0 \geq 1$ and $x_2 \geq 0$, we consider two subcases. If $G(\boldsymbol{x}+\boldsymbol{e_3}) \geq 0$, by h9) we have

$$\begin{aligned}
&[Lv(\boldsymbol{x}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}+\boldsymbol{e_3})] - [Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})] \\
&\leq \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})) - (v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1}))] + \tau^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})) \\
&\quad - (v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}))] + u^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_3})) - (v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x}))] \\
&\leq (1-\alpha)(q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2) < (q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2).
\end{aligned}$$

If $G(\boldsymbol{x}+\boldsymbol{e_3}) < 0$, by h9) we have

$$\begin{aligned}
&[Lv(\boldsymbol{x}+2\boldsymbol{e_3}) - Lv(\boldsymbol{x}+\boldsymbol{e_3})] - [Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x})] \\
&\leq \lambda[(v(\boldsymbol{x}+\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})) - (v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1}))] \\
&\quad + q_1u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3})) - (v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}))] \\
&\quad + q_2u^{-1}[(v(\boldsymbol{x}-\boldsymbol{e_1}+3\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3})) - (v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3}))] \\
&\quad + \tau^{-1}[(v(\boldsymbol{x}+2\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_3})) - (v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x}))] \\
&\leq (1-\alpha)(q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2) < (q_2u^{-1} - \tau^{-1})\tau^2(r_0 - r_2).
\end{aligned}$$

Now, we show that h10) is preserved by considering two separate cases. When $\boldsymbol{x}=(x_0,0,0)$ with $x_0\ge 2$, from (EC.6) we have

$$LG(\boldsymbol{x})$$
$$\le \lambda G(\boldsymbol{x}+\boldsymbol{e_1})+u^{-1}\max\{G(\boldsymbol{x}),0\}+\tau^{-1}\max\{G(\boldsymbol{x}-\boldsymbol{e_1}),0\}+q_1(\tau u)^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1})-v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2})]+r_0\tau^{-1}$$
$$\le (\lambda+u^{-1}+\tau^{-1})r_0+\tau^{-1}(r_0-q_1u^{-1}[v(\boldsymbol{x}-2\boldsymbol{e_1}+\boldsymbol{e_2})-v(\boldsymbol{x}-\boldsymbol{e_1})])\le(1-\alpha)r_0<r_0,$$

where the second inequality holds by h10) and the third inequality holds by h7) and the condition that $u<\tilde{u}(\alpha)$. When $\boldsymbol{x}=\boldsymbol{e_1}$, from (EC.8) we have

$$LG(\boldsymbol{e_1})\le\lambda G(2\boldsymbol{e_1})+u^{-1}\max\{G(\boldsymbol{e_1}),0\}+r_0\tau^{-1}\le(\lambda+u^{-1}+\tau^{-1})r_0=(1-\alpha)r_0<r_0,$$

where the second inequality holds by h10).

Next, we show that h11) is preserved under L by considering four cases. When $x_0\ge 0,\ x_1\ge 1$, and $x_2\ge 0$, we have

$$Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})$$
$$=\lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_2}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_2})]+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]+r_2$$
$$\le\lambda r_2\alpha^{-1}[1-\beta_1^{x_1+x_2+1}\beta_2^{x_0+1}]+\tau^{-1}r_2\alpha^{-1}[1-\beta_1^{x_1+x_2}\beta_2^{x_0}]+u^{-1}r_2\alpha^{-1}[1-\beta_1^{x_1+x_2+1}\beta_2^{x_0}]+r_2$$
$$=r_2\alpha^{-1}[1-\beta_1^{x_1+x_2+1}\beta_2^{x_0}]+r_2\alpha^{-1}\beta_1^{x_1+x_2}\beta_2^{x_0}[\lambda\beta_1(1-\beta_2)-\tau^{-1}(1-\beta_1)+\alpha\beta_1]=r_2\alpha^{-1}[1-\beta_1^{x_1+x_2+1}\beta_2^{x_0}],$$

where the inequality holds by h11) and (EC.3). When $\boldsymbol{x}=\boldsymbol{0}$, by h11) we have

$$Lv(\boldsymbol{e_3})-Lv(\boldsymbol{0})=\lambda[v(\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{e_1})]+u^{-1}[v(\boldsymbol{e_3})-v(\boldsymbol{0})]+r_2\lambda r_2\alpha^{-1}(1-\beta_1\beta_2)+u^{-1}r_2\alpha^{-1}(1-\beta_1)+r_2$$
$$=r_2\alpha^{-1}(1-\beta_1)+r_2\alpha^{-1}[\lambda\beta_1(1-\beta_2)-\tau^{-1}(1-\beta_1)+\alpha\beta_1]=r_2\alpha^{-1}(1-\beta_1),$$

where the last equality holds by (EC.3). When $\boldsymbol{x}=(0,0,x_2)$ with $x_2\ge 1$, by h11) we have

$$Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})=\ \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+\tau^{-1}[v(\boldsymbol{x})-v(\boldsymbol{x}-\boldsymbol{e_3})]+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]+r_2$$
$$\le\lambda r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2]+\tau^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2}]+u^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+1}]+r_2$$
$$=r_2\alpha^{-1}[1-\beta_1^{x_2+1}]+r_2\alpha^{-1}\beta_1^{x_2}[\lambda\beta_1(1-\beta_2)-\tau^{-1}(1-\beta_1)+\alpha\beta_1]=r_2\alpha^{-1}(1-\beta_1^{x_2+1}),$$

where the last equality holds by (EC.3). When $\boldsymbol{x}=(x_0,0,x_2)$ with $x_0\ge 1$ and $x_2\ge 0$, we have

$$\begin{aligned}Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})&=\lambda v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})+\min\{G(\boldsymbol{x}+\boldsymbol{e_3}),0\}+\tau^{-1}v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})+u^{-1}v(\boldsymbol{x}+\boldsymbol{e_3})\\&\quad+r_2-\lambda v(\boldsymbol{x}+\boldsymbol{e_1})-\min\{G(\boldsymbol{x}),0\}-\tau^{-1}v(\boldsymbol{x}-\boldsymbol{e_1})-u^{-1}v(\boldsymbol{x}).\end{aligned}\qquad\text{(EC.11)}$$

We consider two separate cases. If $G(\boldsymbol{x})\ge 0$, then from (EC.11), we have

$$Lv(\boldsymbol{x}+\boldsymbol{e_3})-Lv(\boldsymbol{x})$$
$$\le\lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}+\boldsymbol{e_1})]+\tau^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})-v(\boldsymbol{x}-\boldsymbol{e_1})]+u^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3})-v(\boldsymbol{x})]+r_2$$
$$\le\lambda r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0+1}]+\tau^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0-1}]+u^{-1}r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}]+r_2$$
$$=r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}]+r_2\alpha^{-1}\beta_1^{x_2+1}\beta_2^{x_0-1}[\lambda\beta_2(1-\beta_2)-\tau^{-1}(1-\beta_2)+\alpha\beta_2]\le r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}],$$

where the second inequality holds by h11) and the last inequality holds by (EC.2) and (EC.4). If $G(\boldsymbol{x}) < 0$, then from (EC.11), we have

$$
\begin{aligned}
&Lv(\boldsymbol{x}+\boldsymbol{e_3}) - Lv(\boldsymbol{x}) \le \lambda[v(\boldsymbol{x}+\boldsymbol{e_1}+\boldsymbol{e_3}) - v(\boldsymbol{x}+\boldsymbol{e_1})] + q_1 u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_2})] \\
&\quad + q_2 u^{-1}[v(\boldsymbol{x}-\boldsymbol{e_1}+2\boldsymbol{e_3}) - v(\boldsymbol{x}-\boldsymbol{e_1}+\boldsymbol{e_3})] + \tau^{-1}[v(\boldsymbol{x}+\boldsymbol{e_3}) - v(\boldsymbol{x})] + r_2 \\
&\le \lambda r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0+1}] + q_1 u^{-1} r_2\alpha^{-1}[1-\beta_1^{x_2+2}\beta_2^{x_0-1}] + q_2 u^{-1} r_2\alpha^{-1}[1-\beta_1^{x_2+2}\beta_2^{x_0-1}] \\
&\quad + \tau^{-1} r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}] + r_2 \\
&= r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}] + r_2\alpha^{-1}\beta_1^{x_2+1}\beta_2^{x_0-1}[\lambda\beta_2(1-\beta_2) - u^{-1}(\beta_1-\beta_2) + \alpha\beta_2] \le r_2\alpha^{-1}[1-\beta_1^{x_2+1}\beta_2^{x_0}],
\end{aligned}
$$

where the second inequality holds by h11) and the last inequality holds by (EC.2) and (EC.5).

We prove part (ii) by verifying the conditions in Theorem 6.11.3 of Puterman (2005). It is obvious that the state space $\mathcal{S}$ is countable and we have already verified Assumptions 6.10.1 and 6.10.2 of Puterman (2005) in our proof of Proposition 4. Hence, we only need to show that conditions (a), (b), and (c) in Theorem 6.11.3 of Puterman (2005) hold. Condition (a) holds by part (i). Next, consider a stationary policy π that serves a class 1 customer if $x_1 \ge 1$ and serves a class 2 customer only if $x_0 = x_1 = 0$. Furthermore, under π, if the server triages a class 0 customer in $(x_0, 0, x_2)$ for $x_0 \ge 1$ and $x_2 \ge 0$, then the server will perform triage in $(x_0', 0, x_2')$ where $x_0' \ge x_0$ and $0 \le x_2' \le x_2$. From the optimality equations, we find that $v \in \mathbb{H}$ implies that policy π is an optimal policy. Hence, condition (b) holds. Finally, condition (c) holds, i.e., $\mathbb{H}$ is closed, because the limit of any convergent sequence of functions that satisfy h1) through h11) will satisfy them as well. Hence, there exists an optimal stationary policy whose value function belongs to $\mathbb{H}$. □

References

Puterman, Martin L. 2005. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. Wiley-Interscience.