Electronic Companion for "Pooled versus Dedicated Queues When Customers Are Delay-Sensitive"

Appendix A: Proofs of Lemmas 1 and 2 and a Supplementary Result

Proof of Lemma 1: Consider the pooled system described in Section 2. If there are already $n \geq N$ customers in the system, the expected time that an arriving customer spends in the system is $\bar{W}_p(n+1) = (n+1)/(N\mu)$. The reason is as follows. The arriving customer enters the service after $(n-N+1)$ customers ahead of her are served. Then, because there are N servers, it follows from standard probability arguments that the total expected time to complete the service of $(n-N+1)$ customers ahead of the arriving customer is $(n-N+1)/(N\mu)$. This gives the expected waiting time of the arriving customer in the queue. In addition to this, the expected service time of the aforementioned customer is $1/\mu$. Combining the expected waiting time in the queue and the expected service time, the expected time that the arriving customer spends in the system is $(n-N+1)/(N\mu)+1/\mu = (n+1)/(N\mu)$. This and (2) imply that such an arriving customer joins the system if and only if $R-((n+1)c/(N\mu)) \geq 0$ which is equivalent to $n \leq (RN\mu)/c-1$. This implies that the maximum number customers in the system is equal to K where K is as defined in (7). To find the average sojourn time, we first identify the steady-state probability distribution of the number of customers in the system. Let π_i be the steady-state probability that there are i customers in the system. Because $K \geq N$ by (5), the balance equations of this system are the following: $N\lambda\pi_i = (i+1)\mu\pi_{i+1}$ for $i = 0,\ldots,N-1$ and $\pi_i\lambda = \pi_{i+1}\mu$ for $i = N,\ldots,K-1$. From these and the fact that $\sum_{i=0}^{K}\pi_i = 1$, we have

$$\pi_0 = \left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)^{-1}, \tag{EC.1}$$

$$\pi_i = \pi_0 N^i\rho^i/i! \quad \text{for } i = 1,\ldots,N \quad \text{and} \quad \pi_i = \pi_0 N^N\rho^i/N! \quad \text{for } i = N+1,\ldots,K. \tag{EC.2}$$

Using these, we get the following expressions for the long-run average number of customers in the system and the throughput, respectively:

$$L_p = \sum_{i=0}^{K}\pi_i i = \frac{\sum_{i=0}^{N-1}\frac{N^i}{i!}i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}i\rho^i}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i} \tag{EC.3}$$

$$\lambda_{e,p} = (1-\pi_K)N\lambda = \left(1 - \frac{\frac{N^N}{N!}\rho^K}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i}\right)N\lambda. \tag{EC.4}$$

Because $W_p = L_p/\lambda_{e,p}$ by Little's law, (8) immediately follows from (EC.3) and (EC.4). Replacing (EC.3) and (EC.4) in place of L_p and $\lambda_{e,p}$, respectively, in the SW_p formula (3), we get (9). □

Proof of Lemma 2: Recall that the dedicated system consists of N separate sub-systems each with a line dedicated to one server. Suppose that a customer arrives to one of these dedicated queues and observes that there are n customers in that sub-system. Then, the expected time the arriving customer spends in that sub-system is $\bar{W}_d(n+1) = (n+1)/\mu$. This means by (2) that a customer joins the dedicated queue if and only if $R-(n+1)c/\mu \geq 0$ which is equivalent to $n \leq (R\mu)/c-1$. Note that the later inequality implies that the maximum number of customers in each separate sub-system is k where k is as defined in (5). As a result, each separate sub-system can be considered as an M/M/1/k system. Then, the long-run average number of customers and the throughput in one of the dedicated sub-systems are as follows, respectively (see Table 4 on page 149 of Sztrik (2012)).

$$L_d = \begin{cases} \frac{\rho[1-(k+1)\rho^k+k\rho^{k+1}]}{(1-\rho)(1-\rho^{k+1})} = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} & \text{if } \rho \neq 1 \\ \frac{k}{2} & \text{if } \rho = 1. \end{cases} \tag{EC.5}$$

$$\lambda_{e,d} = \begin{cases} \lambda\left(1 - \frac{(1-\rho)\rho^k}{1-\rho^{k+1}}\right) = \lambda\left(\frac{1-\rho^k}{1-\rho^{k+1}}\right) & \text{if } \rho \neq 1 \\ \lambda\left(\frac{k}{k+1}\right) & \text{if } \rho = 1. \end{cases} \tag{EC.6}$$

By Little's Law, $W_d = L_d/\lambda_{e,d}$. From this, (EC.5) and (EC.6), we get (EC.7) below. Similarly, by substituting L_d and $\lambda_{e,d}$ respectively with (EC.5) and (EC.6) in the SW_d formula (3), we get (EC.8) below.

$$W_d = \begin{cases} \frac{\rho-(k+1)\rho^{k+1}+k\rho^{k+2}}{\lambda(1-\rho)(1-\rho^k)} & \text{if } \rho \neq 1 \\ \frac{k+1}{2\lambda} & \text{if } \rho = 1, \end{cases} \tag{EC.7}$$

$$SW_d = \begin{cases} \left(\frac{1-\rho^k}{1-\rho^{k+1}}\right) RN\lambda - \left(\frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}\right) Nc & \text{if } \rho \neq 1 \\ \frac{k}{k+1} RN\lambda - \frac{k}{2} Nc & \text{if } \rho = 1. \end{cases} \tag{EC.8}$$

The fact that (EC.7) and (EC.8) are equivalent to W_d and SW_d expressions in the lemma, respectively, completes the proof. □

We now state and prove a supplementary lemma which we will use in the remainder of the Appendix.

LEMMA EC.1. *Consider an M/M/1/K queueing system (indexed by $j = s$) with the potential arrival rate λN and the service rate μN and K is as defined in* (7). *Suppose that there is no service fee as in Section 2. Then,* (a) *the throughput is*

$$\lambda_{e,s} = \begin{cases} N\lambda\left(\frac{1-\rho^K}{1-\rho^{K+1}}\right) & \textit{if } \rho \neq 1 \\ N\lambda\frac{K}{K+1} & \textit{if } \rho = 1. \end{cases} \tag{EC.9}$$

(b) *The long-run average number of customers in the system is*

$$L_s = \begin{cases} \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} & \textit{if } \rho \neq 1 \\ \frac{K}{2} & \textit{if } \rho = 1. \end{cases} \tag{EC.10}$$

(c) *The average sojourn time is*

$$W_s = \begin{cases} \frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)(1-\rho^K)N\lambda} & \textit{if } \rho \neq 1 \\ \frac{K+1}{2N\lambda} & \textit{if } \rho = 1. \end{cases} \tag{EC.11}$$

(d) *The social welfare is*

$$SW_s = \begin{cases} \left(\frac{1-\rho^K}{1-\rho^{K+1}}\right) RN\lambda - \left(\frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}\right) c & \textit{if } \rho \neq 1 \\ \frac{K}{K+1} RN\lambda - \frac{K}{2} c & \textit{if } \rho = 1. \end{cases} \tag{EC.12}$$

Proof of Lemma EC.1: Because the ratio of the potential arrival rate to the service rate in the described system is $\frac{N\lambda}{N\mu} = \rho$, which is same as in each dedicated sub-system (which consists of one dedicated queue and one server), the balance equations for the described M/M/1/K system are the same as the ones for the M/M/1/k system analyzed in the proof of Lemma 2, with the exception that k must be replaced with K. Replacing k with K in $\lambda_{e,d}/\lambda$, we get $(1-\tilde{\pi}_K)$ where $\tilde{\pi}_K$ is the steady-state probability of having K customers in the described M/M/1/K system. From this and the fact that the throughput in M/M/1/K system is equal to $(1-\tilde{\pi}_K)N\lambda$, we complete the proof of part (a). Similarly, replacing k with K in L_d we get L_s in part (b). Because the average sojourn time is $W_s = \frac{L_s}{\lambda_{e,s}}$ by Little's Law, part (c) immediately follows. Finally, by replacing λ with $N\lambda$, μ with $N\mu$ and k with K in SW_d/N (from (EC.8)), we get part (d). □

Appendix B: Proof of Proposition 1

Recall that $\theta_d = N\lambda_{e,d}$ and $\theta_p = \lambda_{e,p}$. Let b_d and b_p denote the balking probabilities in the dedicated and pooled systems, respectively. The proofs of Lemmas 1 and 2 imply that

$$b_d = \bar{\pi}_k = \frac{\rho^k}{1+\rho+\cdots+\rho^k} \quad \text{and} \quad b_p = \pi_K = \frac{\frac{N^N}{N!}\rho^K}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i}, \tag{EC.13}$$

where $\bar{\pi}_k$ is the stationary probability that there are k customers in one of the N sub-systems in the dedicated one. Based on these, observe that regardless of the value of ρ, we have

$$\begin{aligned}
& b_d - b_p \\
=& \frac{\rho^k}{\sum_{i=0}^{k}\rho^i} - \frac{\frac{N^N}{N!}\rho^K}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i} \\
=& \frac{1}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)\sum_{i=0}^{k}\rho^i}\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^{i+k} + \frac{N^N}{N!}\sum_{i=N+k}^{K+k}\rho^i - \frac{N^N}{N!}\sum_{i=K}^{K+k}\rho^i\right) \\
=& \frac{1}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)\sum_{i=0}^{k}\rho^i}\left(\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^{i+k} + \frac{N^N}{N!}\sum_{i=N+k-1}^{K+k}\rho^i - \frac{N^N}{N!}\sum_{i=K}^{K+k}\rho^i\right) && \text{(EC.14)} \\
\geq& \frac{1}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)\sum_{i=0}^{k}\rho^i}\left(\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^{i+k}\right) && \text{(EC.15)} \\
>& 0. && \text{(EC.16)}
\end{aligned}$$

The equation (EC.14) is because $\frac{N^{N-1}}{(N-1)!}\rho^{N+k-1} = \frac{N^N}{N!}\rho^{N+k-1}$. The inequality (EC.15) follows from the fact that $N+k-1 \leq K$ since $N+k-1-K \leq N+k-1-Nk = (N-1)(1-k) \leq 0$. Because $\lambda_{e,p} = (1-b_p)N\lambda$ and $\lambda_{e,d} = (1-b_d)\lambda$, and the balking probability in the dedicated system is strictly larger than the one in the pooled system by (EC.16), we have $\lambda_{e,d}N < \lambda_{e,p}$. □

Appendix C: Proof of Proposition 2

Part (a): Recall $\lambda_{e,p}$ and $\lambda_{e,s}$ from (EC.4) and (EC.9), respectively. Then,

$$\theta_p = \lambda_{e,p} = \left(1 - \frac{\frac{N^N}{N!}\rho^K}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i}\right)N\lambda < \left(1 - \frac{\rho^K}{\sum_{i=0}^{K}\rho^i}\right)N\lambda = \lambda_{e,s} = \theta_s. \tag{EC.17}$$

Part (b): Denote by X_s the number of customers in the SQ system in the steady-state, and let X_p be the corresponding figure in the pooled system. Note that the SQ system is the same as the M/M/1/K system described in Lemma EC.1. To show our claim, we will use the standard likelihood comparison technique. Let $\gamma_s(m+1)$ be the transition rate from state $m+1$ to m in the SQ system, $\gamma_p(m+1)$ be the transition rate from state $m+1$ to m in the pooled system, for any $m = 0, 1, \ldots K-1$. Because $\gamma_p(m+1) \leq \gamma_s(m+1)$ for each m, in the steady-state, we have

$$\mathbb{P}(X_p = m)N\lambda \leq \mathbb{P}(X_p = m+1)\gamma_s(m+1) \quad \text{and} \quad \mathbb{P}(X_s = m)N\lambda = \mathbb{P}(X_s = m+1)\gamma_s(m+1).$$

Thus, we have

$$\frac{\mathbb{P}(X_p=m+1)}{\mathbb{P}(X_p=m)} \geq \frac{N\lambda}{\gamma_s(m+1)} = \frac{\mathbb{P}(X_s=m+1)}{\mathbb{P}(X_s=m)}. \tag{EC.18}$$

Using (EC.18), we now show that $L_p \doteq \mathbb{E}(X_p) \geq L_s \doteq \mathbb{E}(X_s)$. Note that (EC.18) implies that $\frac{\mathbb{P}(X_p=j)}{\mathbb{P}(X_p=i)} \geq \frac{\mathbb{P}(X_s=j)}{\mathbb{P}(X_s=i)}$ for all $i \leq j$, $i,j \in \{0,1,\ldots,K\}$, which is equivalent to

$$\mathbb{P}(X_p=j)\mathbb{P}(X_s=i) \geq \mathbb{P}(X_p=i)\mathbb{P}(X_s=j). \tag{EC.19}$$

The summation on both sides of (EC.19) over i from 0 to j gives

$$\mathbb{P}(X_p=j)\mathbb{P}(X_s\leq j) \geq \mathbb{P}(X_p\leq j)\mathbb{P}(X_s=j). \tag{EC.20}$$

Similarly, the summation on both sides of (EC.19) over j from $i+1$ to K results in

$$\mathbb{P}(X_p\geq i+1)\mathbb{P}(X_s=i) \geq \mathbb{P}(X_p=i)\mathbb{P}(X_s\geq i+1). \tag{EC.21}$$

Combining (EC.20) and (EC.21) and letting $i=j=a$, we have

$$\frac{\mathbb{P}(X_p\geq a+1)}{\mathbb{P}(X_s\geq a+1)} \geq \frac{\mathbb{P}(X_p=a)}{\mathbb{P}(X_s=a)} \geq \frac{\mathbb{P}(X_p\leq a)}{\mathbb{P}(X_s\leq a)}. \tag{EC.22}$$

Thus, $\mathbb{P}(X_p\leq a) \leq \mathbb{P}(X_s\leq a)$ for any $a \in \{0,1,2,\ldots,K\}$, hence

$$L_p = \mathbb{E}(X_p) = \sum_{i=0}^{K}(1-\mathbb{P}(X_p\leq i)) \geq \sum_{i=0}^{K}(1-\mathbb{P}(X_s\leq i)) = \mathbb{E}(X_s) = L_s. \tag{EC.23}$$

By (EC.23), (EC.17) and Little's Law,

$$W_p = \frac{L_p}{\lambda_{e,p}} \geq \frac{L_s}{\lambda_{e,p}} > \frac{L_s}{\lambda_{e,s}} = W_s.$$

Part (c): Recall the definition of social welfare from (3):

$$SW_p = \lambda_{e,p}(R-cW_p) \qquad \text{and} \qquad SW_s = \lambda_{e,s}(R-cW_s).$$

Because $\lambda_{e,s} > \lambda_{e,p}$ and $W_s < W_p$, $SW_s > SW_p$. This completes the proof of part (c). □

Appendix D: Proof of Proposition 3

Recall Lemmas 1, 2 and EC.1. To prove Proposition 3, we shall define some constants. Let

$$\eta \doteq z_1 + 1$$

where

$$z_1 \doteq \inf\left\{z\in\mathbb{R} : z > \frac{1}{\ln(\rho)} - 1, \max\left\{\frac{(z+1)N\rho}{\rho^{z+1}-1} - \frac{N-1}{4(\rho-1)^2}, \frac{(z+1)N}{(N-1)\rho^z} - \frac{1}{2}\right\} < 0\right\}. \tag{EC.24}$$

Part (a): This follows from Propositions 1 and 2-(a).

The outline of the remainder of our proof is as follows. First, we will state and prove Lemma EC.2 that shows the existence of the constant z_1 defined in (EC.24). Based on this, we will prove parts (b) and (c).

LEMMA EC.2. *For $\rho > 1$, the constant z_1 defined in (EC.24) exists and $z_1 \in [1,\infty)$.*

Proof of Lemma EC.2: Define $g_1(z) \doteq \frac{N\rho(z+1)}{\rho^{z+1}-1}$ and $g_2(z) \doteq \frac{N(z+1)}{N-1}\rho^{-z}$. Then, note that the definition in (EC.24) is equivalent to $z_1 \doteq \inf\{z \in \mathbb{R} : g_1(z) < (N-1)/(4(\rho-1)^2), g_2(z) < 1/2 \text{ and } z > 1/\ln(\rho) - 1\}$. Both $g_1(\cdot)$ and $g_2(\cdot)$ are strictly decreasing when $\rho > 1$ and $z > \frac{1}{\ln(\rho)} - 1$ because

$$g_1'(z) = N\rho\frac{\rho^{z+1} - 1 - (z+1)\rho^{z+1}\ln(\rho)}{(\rho^{z+1}-1)^2} = N\rho\frac{\rho^{z+1}\left(1-(z+1)\ln(\rho)\right)-1}{(\rho^{z+1}-1)^2} < 0, \text{ and} \tag{EC.25}$$

$$g_2'(z) = \frac{N}{N-1}\frac{\rho^z - (z+1)\rho^z\ln(\rho)}{\rho^{2z}} = \frac{N}{N-1}\frac{\rho^z\left(1-(z+1)\ln(\rho)\right)}{\rho^{2z}} < 0, \tag{EC.26}$$

for $\rho > 1$ and $z > \frac{1}{ln(\rho)} - 1$. In addition, we have

$$\lim_{z\to\infty} g_1(z) = 0 \qquad \text{and} \qquad \lim_{z\to\infty} g_2(z) = 0. \tag{EC.27}$$

It follows from (EC.25) through (EC.27) that z_1 exists and it is finite. We now show that $z_1 \geq 1$ for $\rho > 1$. Suppose for a contradiction that $z_1 < 1$. Because $g_1(z)$ and $g_2(z)$ are strictly decreasing for $\rho > 1$ and $z > \frac{1}{\ln(\rho)} - 1$, and $z_1 \geq \frac{1}{\ln(\rho)} - 1$ by definition of z_1, at $z = 1$ the following inequalities must hold: $g_1(1) < \frac{N-1}{4(\rho-1)^2}$ and $g_2(1) < \frac{1}{2}$. Note that

$$g_2(1) = \frac{2N}{(N-1)\rho} < \frac{1}{2} \Leftrightarrow \rho > \frac{4N}{N-1}, \tag{EC.28}$$

which implies that $\rho > 4$. Observe also that

$$g_1(1) = \frac{2N\rho}{\rho^2-1} < \frac{N-1}{4(\rho-1)^2} \Leftrightarrow \left(\frac{(\rho-1)2N\rho}{\rho+1} - \frac{N-1}{4}\right)\frac{1}{(\rho-1)^2} < 0. \tag{EC.29}$$

But, for $\rho > 4$,

$$\frac{(\rho-1)2N\rho}{\rho+1} - \frac{N-1}{4} > \frac{24N}{5} - \frac{N-1}{4} > 0,$$

which contradicts (EC.29). Thus, $z_1 \geq 1$. □

<u>Part (b):</u> Proposition 3-(b) follows from the explicit forms of L_d and L_s in the proofs of Lemmas 2 and EC.1, from the fact that $\nu = \frac{R\mu}{c} > \eta = z_1 + 1$ implies $k > z_1$, and from Lemma EC.3, which will be proved below. We now state and verify Lemma EC.3.

LEMMA EC.3. *If $\rho > 1$ and $k > z_1$ where z_1 is defined in* (EC.24)*,*

$$\left(\frac{\rho - (1+K)\rho^{K+1} + K\rho^{K+2}}{(\rho-1)^2(\rho^{K+1}-1)} - N\frac{\rho-(1+k)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)^2(\rho^{k+1}-1)}\right) > \frac{N-1}{2(\rho-1)^2}.$$

Proof of Lemma EC.3: To prove this claim, we first prove that $h(\rho,x) \doteq \frac{\rho-(1+x)\rho^{x+1}+x\rho^{x+2}}{(\rho-1)^2(\rho^{x+1}-1)}$ is increasing in x for $\rho > 1$. Note that

$$h(\rho,x) = \frac{1}{\rho-1}\frac{\rho-(1+x)\rho^{x+1}+x\rho^{x+2}}{(\rho-1)(\rho^{x+1}-1)} = \frac{1}{\rho-1}\frac{\rho-(1+x)\rho^{x+1}+x\rho^{x+2}}{(1-\rho)(1-\rho^{x+1})} = \frac{1}{\rho-1}\left(\frac{\rho}{1-\rho} - \frac{(x+1)\rho^{x+1}}{1-\rho^{x+1}}\right).$$

Then,

$$\frac{\partial h(\rho,x)}{\partial x} = -\frac{\rho^{x+1}}{(\rho-1)(1-\rho^{x+1})^2}(1-\rho^{x+1}+(x+1)\ln(\rho)). \tag{EC.30}$$

Let $v(\rho,x) \doteq 1 - \rho^{x+1} + (x+1)\ln(\rho)$ and observe from (EC.30) that $v(\rho,x)$ and $\frac{\partial h(\rho,x)}{\partial x}$ have opposite signs for $\rho > 1$. Note that $v(1,x) = 0$ for any x, and for $\rho > 1$ and $x \geq 0$,

$$\frac{\partial v(\rho,x)}{\partial \rho} = -(x+1)\rho^x + (x+1)\frac{1}{\rho} = (x+1)\left(-\rho^x + \frac{1}{\rho}\right) < 0.$$

This immediately implies that $\frac{\partial h(\rho,x)}{\partial x} > 0$ for $\rho > 1$ and $x \geq 0$. Based on this, for $\rho > 1$, we have

$$\begin{aligned}
&\frac{\rho-(1+K)\rho^{K+1}+K\rho^{K+2}}{(\rho-1)^2(\rho^{K+1}-1)} - N\frac{\rho-(1+k)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)^2(\rho^{k+1}-1)}\\
\geq&\frac{\rho-(1+Nk)\rho^{Nk+1}+Nk\rho^{Nk+2}}{(\rho-1)^2(\rho^{Nk+1}-1)} - N\frac{\rho-(1+k)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)^2(\rho^{k+1}-1)} \quad \text{(EC.31)}\\
=&\frac{1}{(\rho-1)^2}\frac{(N-1)\rho^{(N+1)k+2}-N(k+1)\rho^{Nk+2}+(Nk+1)\rho^{Nk+1}+\rho^{k+2}-\rho+N(\rho-(1+k)\rho^{k+1}+k\rho^{k+2})}{(\rho^{Nk+1}-1)(\rho^{k+1}-1)}\\
>&\frac{1}{(\rho-1)^2}\frac{(N-1)\rho^{(N+1)k+2}-N(k+1)\rho^{Nk+2}}{(\rho^{Nk+1}-1)(\rho^{k+1}-1)} \quad \text{(EC.32)}\\
=&\frac{1}{(\rho-1)^2}\frac{(N-1)\rho^{(N+1)k+2}-N(k+1)\rho^{Nk+2}}{\rho^{(N+1)k+2}-\rho^{Nk+1}-\rho^{k+1}+1}\\
=&\frac{N-1}{(\rho-1)^2}\frac{1-\frac{N(k+1)}{N-1}\rho^{-k}}{1-\rho^{-k-1}-\rho^{-Nk-1}+\rho^{-(Nk+k+2)}}. \quad \text{(EC.33)}
\end{aligned}$$

We have (EC.31) because $K \geq Nk$ and as we showed earlier, $h(\rho,x) = \frac{\rho-(1+x)\rho^{x+1}+x\rho^{x+2}}{(\rho-1)^2(\rho^{x+1}-1)}$ is increasing in x for $\rho > 1$. The inequality (EC.32) holds because $W_d > 0$ in the first line of (EC.7) implies that $\rho-(1+k)\rho^{k+1}+k\rho^{k+2} > 0$. We already know from the proof of Lemma EC.2 that $g_2(z) = \frac{(z+1)N}{N-1}\rho^{-z}$ is strictly decreasing in z if $z > \frac{1}{\ln(\rho)} - 1$ and $\rho > 1$, and $\lim_{z\to\infty} g_2(z) = 0$. This, the definition of z_1 in (EC.24) and the fact that $(1-\rho^{-k-1}-\rho^{-Nk-1}+\rho^{-(Nk+k+2)}) \in (0,1)$ for $\rho > 1$ imply that

$$\frac{N-1}{(\rho-1)^2}\frac{1-\frac{N(k+1)}{N-1}\rho^{-k}}{1-\rho^{-k-1}-\rho^{-Nk-1}+\rho^{-(Nk+k+2)}} > \frac{N-1}{2(\rho-1)^2} \quad \text{for } k > z_1,\ \rho > 1. \quad \text{(EC.34)}$$

Combining this and (EC.33), it follows that if $\rho > 1$ and $k > z_1$,

$$\frac{\rho-(1+K)\rho^{K+1}+K\rho^{K+2}}{(\rho-1)^2(\rho^{K+1}-1)} - N\frac{\rho-(1+k)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)^2(\rho^{k+1}-1)} > \frac{N-1}{2(\rho-1)^2}. \quad \text{(EC.35)}$$

This completes the proof of our claim. □

Part (c): In this part, we will show that if $\rho > 1$ and $k > z_1$,

$$SW_d - SW_s > c\frac{(N-1)}{(\rho-1)4}. \quad \text{(EC.36)}$$

This completes the proof of social welfare claim in Proposition 3-(c) because $\nu = \frac{R\mu}{c} > \eta \doteq z_1 + 1$ implies $k > z_1$.

Recall from (EC.8) that the social welfare in the dedicated system is

$$\begin{aligned}
SW_d &= \left(\frac{1-\rho^k}{1-\rho^{k+1}}\right)RN\lambda - \left(\frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}\right)Nc\\
&= RN\lambda - \frac{1-\rho}{1-\rho^{k+1}}\left(N\lambda R\rho^k + Nc\frac{\rho-(1+k)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)^2}\right). \quad \text{(EC.37)}
\end{aligned}$$

Recall also from (EC.12) that

$$\begin{aligned}
SW_s &= \left(\frac{1-\rho^K}{1-\rho^{K+1}}\right)RN\lambda - \left(\frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}\right)c\\
&= RN\lambda - \frac{1-\rho}{1-\rho^{K+1}}\left(N\lambda R\rho^K + c\frac{\rho-(1+K)\rho^{K+1}+K\rho^{K+2}}{(\rho-1)^2}\right). \quad \text{(EC.38)}
\end{aligned}$$

From (EC.37) and (EC.38), it follows that $SW_d - SW_s > c\frac{(N-1)}{(\rho-1)4}$ if and only if

$$\begin{aligned}
&\frac{1}{\rho^{k+1}-1}\left(RN\lambda\rho^k + Nc\frac{\rho-(1+k)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)^2}\right)\\
&\qquad - \frac{1}{\rho^{K+1}-1}\left(RN\lambda\rho^K + c\frac{\rho-(1+K)\rho^{K+1}+K\rho^{K+2}}{(\rho-1)^2}\right) < -c\frac{(N-1)}{4(\rho-1)^2}.
\end{aligned}$$

Note that the left hand side of the above inequality is equivalent to

$$RN\lambda\left(\frac{\rho^k}{\rho^{k+1}-1}-\frac{\rho^K}{\rho^{K+1}-1}\right)-c\left(\frac{\rho-(1+K)\rho^{K+1}+K\rho^{K+2}}{(\rho-1)^2(\rho^{K+1}-1)}-N\frac{\rho-(1+k)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)^2(\rho^{k+1}-1)}\right).$$

Rearranging this, we conclude that $SW_d - SW_s > c\frac{(N-1)}{(\rho-1)4}$ if and only if

$$\underbrace{N\lambda\frac{R}{c}\left(\frac{\rho^k}{\rho^{k+1}-1}-\frac{\rho^K}{\rho^{K+1}-1}\right)}_{\text{First Term}}-\underbrace{\left(\frac{\rho-(1+K)\rho^{K+1}+K\rho^{K+2}}{(\rho-1)^2(\rho^{K+1}-1)}-N\frac{\rho-(1+k)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)^2(\rho^{k+1}-1)}\right)}_{\text{Second Term}}$$
$$<-\frac{(N-1)}{4(\rho-1)^2} \tag{EC.39}$$

We claim and show below that the first term in (EC.39) is bounded above by $(N-1)/(4(\rho-1)^2)$ if $\rho>1$ and $k>z_1$. Moreover, by Lemma EC.3, the second term in (EC.39) is bounded below by $(N-1)/(2(\rho-1)^2)$ if $\rho>1$ and $k>z_1$. These two results imply that if $\rho>1$ and $k>z_1$, we have (EC.39). From this, Proposition 3-(c) follows.

We now show our claim that if $\rho>1$ and $k>z_1$, the first term in (EC.39) is bounded above by $(N-1)/(4(\rho-1)^2)$. Suppose that $\rho>1$. Then, the first term in (EC.39) satisfies the following inequality:

$$N\lambda\frac{R}{c}\left(\frac{\rho^k}{\rho^{k+1}-1}-\frac{\rho^K}{\rho^{K+1}-1}\right)<N\lambda\frac{k+1}{\mu}\left(\frac{\rho^k}{\rho^{k+1}-1}-\frac{\rho^K}{\rho^{K+1}-1}\right) \tag{EC.40}$$
$$<N\lambda\frac{k+1}{\mu}\left(\frac{\rho^k}{\rho^{k+1}-1}-\frac{\rho^{Nk+N}}{\rho^{Nk+N+1}-1}\right) \tag{EC.41}$$
$$=N\rho(k+1)\frac{\rho^{Nk+N}-\rho^k}{(\rho^{k+1}-1)(\rho^{Nk+N+1}-1)} \tag{EC.42}$$
$$<\frac{N\rho(k+1)}{(\rho^{k+1}-1)}. \tag{EC.43}$$

The inequality (EC.40) above follows from the definition of k in (5) and the fact that the first term in (EC.39) is positive. Note from the definitions of k and K in (5) and (7) that we also have $K<Nk+N$. Then, the inequality (EC.41) follows because $K<Nk+N$ and $\frac{\rho^K}{\rho^{K+1}-1}$ is decreasing in K for $\rho>1$. The inequality (EC.43) is because $\rho^{Nk+N}-\rho^k<\rho^{Nk+N+1}-1$ for $\rho>1$.

We already know from the proof of Lemma EC.2 that $g_1(z)\doteq\frac{(z+1)N\rho}{(\rho^{z+1}-1)}$ is strictly decreasing in z for $\rho>1$ and $z>\frac{1}{\ln(\rho)}-1$. Recall the definition of z_1 in (EC.24). Because $z_1\geq\frac{1}{\ln(\rho)}-1$ and $\frac{(z_1+1)N\rho}{(\rho^{z_1+1}-1)}\leq\frac{N-1}{4(\rho-1)^2}$, we have the following for $k>z_1$:

$$\frac{N\rho(k+1)}{(\rho^{k+1}-1)}<\frac{N-1}{4(\rho-1)^2},$$

which together with (EC.43) implies that

$$N\lambda\frac{R}{c}\left(\frac{\rho^k}{\rho^{k+1}-1}-\frac{\rho^K}{\rho^{K+1}-1}\right)<\frac{N-1}{4(\rho-1)^2}. \tag{EC.44}$$

This completes our arguments for the proof of our claim above that the first term in (EC.39) is bounded above by $(N-1)/(4(\rho-1)^2)$ if $\rho>1$ and $k>z_1$ and hence completes our arguments for the proof of Proposition 3-(c). □

Proof of the claim that $W_s>W_d$ when (12) in Remark 1-(a): Recall that $SW_d=\lambda_{e,d}N(R-cW_d)$ and $SW_s=\lambda_{e,s}(R-cW_s)$. By Proposition 3-(a), we have

$$\lambda_{e,d}N<\lambda_{e,s}. \tag{EC.45}$$

From this and Proposition 3-(c), the claim immediately follows. □

Appendix E: The Statement and the Proof of Proposition EC.1

PROPOSITION EC.1. *The dedicated system results in* **(i)** *strictly larger average sojourn time and* **(ii)** *strictly smaller social welfare than the SQ system, i.e., $W_d > W_s$ and $SW_d < SW_s$, respectively, if*

$$\rho < 1 \quad \text{and} \quad \nu > \widetilde{\eta} \doteq z_2 + 1 \tag{EC.46}$$

where

$$z_2 \doteq \inf\left\{z \in \mathbb{R} : z > -1/\ln(\rho), z\rho^z < (N-1)/N\right\}. \tag{EC.47}$$

Proof of Proposition EC.1: We first prove a lemma which will be used in the remainder of the proof.

LEMMA EC.4. *For $\rho < 1$, the constant z_2 defined in $(EC.47)$ exists and it is finite.*

Proof of Lemma EC.4: Define $g_3(z) \doteq z\rho^z$. Then, note that the definition in (EC.47) is equivalent to $z_2 \doteq \inf\{z \in \mathbb{R} : g_3(z) < (N-1)/N,$ and $z > -1/\ln(\rho)\}$. The function $g_3(\cdot)$ is strictly decreasing when $\rho < 1$ and $z > -\frac{1}{\ln(\rho)}$ because

$$g_3'(z) = \rho^z + z\rho^z \ln(\rho) = \rho^z(1 + z\ln(\rho)) < 0, \tag{EC.48}$$

for $\rho < 1$ and $z > -\frac{1}{\ln(\rho)}$. In addition, we have

$$\lim_{z\to\infty} g_3(z) = 0. \tag{EC.49}$$

It follows from (EC.48) and (EC.49) that z_2 exists and it is finite. □

Recall from (EC.7) that when $\rho < 1$, the average sojourn time in the dedicated system is

$$W_d = \frac{\rho - (k+1)\rho^{k+1} + k\rho^{k+2}}{\lambda(\rho^k - 1)(\rho - 1)}.$$

Recall from (EC.11) that when $\rho < 1$, the average sojourn time in the SQ system is

$$W_s = \frac{1}{N\lambda(\rho-1)} \frac{K\rho^{K+2} - (K+1)\rho^{K+1} + \rho}{\rho^K - 1}.$$

If $\rho < 1$ and $k > z_2$,

$$\begin{aligned}
W_d - W_s =& \frac{\rho - (k+1)\rho^{k+1} + k\rho^{k+2}}{\lambda(\rho^k - 1)(\rho - 1)} - \frac{1}{N\lambda(\rho-1)} \frac{K\rho^{K+2} - (K+1)\rho^{K+1} + \rho}{\rho^K - 1} \\
\geq& \frac{\rho - (k+1)\rho^{k+1} + k\rho^{k+2}}{\lambda(\rho^k - 1)(\rho - 1)} - \frac{1}{N\lambda(\rho-1)} \frac{(Nk+N-1)\rho^{Nk+N+1} - (Nk+N)\rho^{Nk+N} + \rho}{\rho^{Nk+N-1} - 1} \qquad \text{(EC.50)} \\
=& \big((\rho - (k+1)\rho^{k+1} + k\rho^{k+2})N(1-\rho^{Nk+N-1}) - ((Nk+N-1)\rho^{Nk+N+1} - (Nk+N)\rho^{Nk+N} + \rho)(1-\rho^k)\big) \\
& \frac{1}{(1-\rho)(1-\rho^k)(1-\rho^{Nk+N-1})N\lambda} \\
=& \big((N-1)\rho^{Nk+k+N+1} - (Nk+N-1)\rho^{Nk+N+1} + Nk\rho^{Nk+N} + Nk\rho^{k+2} - (Nk+N-1)\rho^{k+1} + (N-1)\rho\big) \\
& \frac{1}{(1-\rho)(1-\rho^k)(1-\rho^{Nk+N-1})N\lambda} \\
=& \big((N-1)(\rho^{Nk+k+N+1} - \rho^{Nk+N+1} - \rho^{k+1} + \rho) + Nk(\rho^{Nk+N} - \rho^{Nk+N+1} + \rho^{k+2} - \rho^{k+1})\big) \\
& \frac{1}{(1-\rho)(1-\rho^k)(1-\rho^{Nk+N-1})N\lambda} \\
=& \big((N-1)(\rho - \rho^{Nk+N+1})(1-\rho^k) - Nk(\rho^{k+1} - \rho^{Nk+N})(1-\rho)\big) \frac{1}{(1-\rho)(1-\rho^k)(1-\rho^{Nk+N-1})N\lambda} \\
>& \big((N-1)(\rho - \rho^{Nk+N+1}) - Nk(\rho^{k+1} - \rho^{Nk+N})\big) \frac{1}{(1-\rho^k)(1-\rho^{Nk+N-1})N\lambda} \\
>& \big((N-1)\rho^{-k} - Nk\big) \frac{\rho^{k+1} - \rho^{Nk+N}}{(1-\rho^k)(1-\rho^{Nk+N-1})N\lambda} \qquad \text{(EC.51)} \\
>& 0, \qquad \text{(EC.52)}
\end{aligned}$$

which proves the claim in part (i).

We now explain why the inequalities in (EC.50) and (EC.52) hold. Note that W_s is increasing in K because

$$
\begin{aligned}
&\frac{\partial\left(\frac{1}{N\lambda(\rho-1)}\frac{K\rho^{K+2}-(K+1)\rho^{K+1}+\rho}{\rho^K-1}\right)}{\partial K}\\
=&\frac{1}{N\lambda(\rho-1)}\frac{(\rho^{K+2}+K\rho^{K+2}\ln(\rho)-\rho^{K+1}-(K+1)\rho^{K+1}\ln(\rho))(\rho^K-1)-(K\rho^{K+2}-(K+1)\rho^{K+1}+\rho)(\rho^K\ln(\rho))}{(\rho^K-1)^2}\\
=&\frac{1}{N\lambda(\rho-1)}\frac{-\rho^{K+2}-K\rho^{K+2}\ln(\rho)+\rho^{K+1}+K\rho^{K+1}\ln(\rho)+\rho^{2K+2}-\rho^{2K+1}}{(\rho^K-1)^2}\\
=&\frac{1}{N\lambda(\rho-1)}\cdot\frac{(\rho-1)\rho^{K+1}(\rho^K-K\ln(\rho)-1)}{(\rho^K-1)^2}\\
=&\frac{1}{N\lambda}\cdot\frac{\rho^{K+1}(\rho^K-K\ln(\rho)-1)}{(\rho^K-1)^2}\\
\geq&0. \qquad \text{(EC.53)}
\end{aligned}
$$

(Here, (EC.53) follows from the fact that $\rho^K-K\ln(\rho)-1\geq 0$, which is because $\frac{\partial(\rho^K-K\ln(\rho)-1)}{\partial\rho}=K\rho^{K-1}-\frac{K}{\rho}=\frac{K}{\rho}(\rho^K-1)$ and thus $\rho^K-K\ln(\rho)-1$ achieves the minimum, which is 0, at $\rho=1$.) Then, (EC.50) follows from the fact that $K\leq Nk+N-1$ and W_s is increasing in K, as shown above.

The reason for (EC.52) is as follows. We already know from the proof of Lemma EC.4 that $g_3(z)\doteq z\rho^z$ is strictly decreasing in z for $\rho<1$ and $z>-\frac{1}{\ln(\rho)}$. Recall the definition of z_2 in (EC.47). Because $z_2\geq-\frac{1}{\ln(\rho)}$ and $z_2\rho^{z_2}\leq\frac{N-1}{N}$, we have $k\rho^k<\frac{N-1}{N}$ for $\rho<1$ and $k>z_2$.

We now prove the claim in part (ii). Recall from (3) that

$$SW_d=\lambda_{e,d}N(R-cW_d)\quad\text{and}\quad SW_s=\lambda_{e,s}(R-cW_s).$$

Since $\lambda_{e,d}N<\lambda_{e,s}$ by Proposition 3-(a), part (i) implies part (ii). □

Appendix F: Proof of Theorem 1

F.1. Proof of Theorem 1 - Part (a):

Part (a)-(i) follows from Propositions 2-(b) and Remark 1-(a). Part (a)-(ii) follows from Propositions 2-(c) and 3-(c). □

F.2. Proof of Theorem 1 - Part (b):

We will first state and prove a lemma. Using this lemma, we will then prove parts (i) and (ii) under the condition (14), and then we will prove same claims under the condition (15).

Proof of Theorem 1-(b) under the condition (14)**:** If $\nu<\frac{N+1}{N}$ which is equivalent to $\frac{R}{c}<\frac{N+1}{N\mu}$, then $\frac{R\mu}{c}<\frac{N+1}{N}$ and $\frac{RN\mu}{c}<N+1$. This and (5) imply that $k=1$ and $K=N$. Thus, under (14), there is no waiting line and a joining customer immediately gets the service. As a result,

$$W_d=W_p=\frac{1}{\mu},$$

which is the claim in part (i). Recall the definition of the social welfare from (3):

$$SW_d=\lambda_{e,d}N(R-cW_d)=\lambda_{e,d}N\left(R-\frac{c}{\mu}\right)\quad\text{and}\quad SW_p=\lambda_{e,p}(R-cW_p)=\lambda_{e,p}\left(R-\frac{c}{\mu}\right).$$

Based on this, because $\lambda_{e,d}N<\lambda_{e,p}$ by Proposition 1, $SW_d<SW_p$. □

Proof of Theorem 1-(b) under the condition (15)**:** Define

$$\bar{\eta} \doteq z_3 + 1, \tag{EC.54}$$

where

$$z_3 \doteq \inf\left\{z \in \mathbb{R} : z > -(\ln(\rho)^{-1} + 3) \text{ and } (z+3)\rho^{z-1} < (N-1)/N\right\}. \tag{EC.55}$$

In light of this, the outline of the remainder of the proof is as follows. First, we will state and prove Lemma EC.5 that shows the existence of the constant z_3 which will be used in the remainder of the proof. Then, we will show in Lemma EC.6 that if $\rho < 1$ and $k > z_3$, we have $L_p - NL_d < 0$. Finally, we will use this inequality to prove the claims in parts (i) and (ii) for $\rho < 1$ and $k > z_3$. This and the fact that $\nu = \frac{R\mu}{c} > z_3 + 1$ implies $k > z_3$ complete the proof of part (b).

LEMMA EC.5. *For $\rho < 1$, the constant z_3 defined in $(EC.55)$ exists and it is finite.*

Proof of Lemma EC.5: Define $g_4(z) \doteq (z+3)\rho^{z-1}$. Then, note that the definition in (EC.55) is equivalent to $z_3 \doteq \inf\{z \in \mathbb{R} : g_4(z) < (N-1)/N \text{ and } z > -1/\ln(\rho) - 3\}$. Observe that $g_4(\cdot)$ is strictly decreasing for $\rho < 1$ and $z > -\frac{1}{\ln(\rho)} - 3$ because

$$g_4'(z) = \rho^{z-1} + (z+3)\rho^{z-1}\ln(\rho) = \rho^{z-1}(1 + (z+3)\ln(\rho)) < 0. \tag{EC.56}$$

In addition, by an application of L'Hopital's Rule, we have

$$\lim_{z\to\infty} g_4(z) = \lim_{z\to\infty}(z+3)\rho^{z-1} = 0. \tag{EC.57}$$

From (EC.56) and (EC.57), the claim follows. □

LEMMA EC.6. *For $\rho < 1$ and $k > z_3$, the long-run average number of customers in the dedicated system, that is, NL_d, and the long-run average number of customers in the pooled system satisfy the following inequality:*

$$L_p - NL_d < 0. \tag{EC.58}$$

Proof of Lemma EC.6: Recall L_p and L_d from (EC.3) and (EC.5). Then, we have

$$\begin{aligned}
&L_p - NL_d \\
&= \frac{\sum_{i=0}^{N-1}\frac{N^i}{i!}i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K} i\rho^i}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i} - N\frac{\rho\,(1-(k+1)\rho^k + k\rho^{k+1})}{(1-\rho)(1-\rho^{k+1})} \\
&= \frac{\left[\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K} i\rho^i\right)(1-\rho)(1-\rho^{k+1}) - N\rho\,(1-(k+1)\rho^k + k\rho^{k+1})\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})}
\end{aligned} \tag{EC.59}$$

Note that we have

$$\sum_{i=0}^{N-1}\frac{N^i}{i!}i\rho^i = \sum_{i=1}^{N-1}\frac{N^i}{i!}i\rho^i = \sum_{i=1}^{N-1}\frac{N^i}{(i-1)!}\rho^i = N\rho\sum_{i=1}^{N-1}\frac{N^{i-1}}{(i-1)!}\rho^{i-1} = N\rho\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i,$$

$$\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i = \sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N-1}^{K}\rho^i = \sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\frac{\rho^{N-1}-\rho^{K+1}}{1-\rho}.$$

Thus, (EC.59) and $L_p - NL_d$ are equivalent to

$$
\begin{aligned}
&= \frac{\left(N\rho \sum_{i=0}^{N-2} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K} i\rho^i\right)(1-\rho)(1-\rho^{k+1})}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})} \\
&- \frac{N\rho\left(1-(k+1)\rho^k + k\rho^{k+1}\right)\left(\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\frac{\rho^{N-1}-\rho^{K+1}}{1-\rho}\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})} \\
&= \frac{\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i\left(N\rho(1-\rho)(1-\rho^{k+1}) - N\rho\left(1-(k+1)\rho^k+k\rho^{k+1}\right)\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})} \\
&+ \frac{\frac{N^N}{N!}\left(\frac{-(K+1)\rho^{K+1}+K\rho^{K+2}+N\rho^N-(N-1)\rho^{N+1}}{(1-\rho)^2}(1-\rho)(1-\rho^{k+1}) - N\rho\left(1-(k+1)\rho^k+k\rho^{k+1}\right)\frac{\rho^{N-1}-\rho^{K+1}}{1-\rho}\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})}
\end{aligned} \tag{EC.60}
$$

$$
\begin{aligned}
&= \frac{\left(-\rho+(k+1)\rho^k-(k+1)\rho^{k+1}+\rho^{k+2}\right)N\rho\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i}{(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i)(1-\rho)(1-\rho^{k+1})} \\
&+ \frac{(N^N/N!)\left[-(N-1)\rho^{N+1}-(K+1)\rho^{K+1}-N(k+1)\rho^{N+k+1}+N(k+1)\rho^{N+k}+(N-1)\rho^{N+k+2}\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)^2(1-\rho^{k+1})} \\
&+ \frac{(N^N/N!)\left[(K+N)\rho^{K+2}+(K+1-Nk-N)\rho^{K+k+2}+(Nk-K)\rho^{K+k+3}\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)^2(1-\rho^{k+1})}.
\end{aligned} \tag{EC.61}
$$

Equation (EC.60) holds because

$$
\begin{aligned}
\sum_{i=N}^{K} i\rho^i &= \rho\frac{\partial}{\partial\rho}\left(\rho^N+\rho^{N+1}+\ldots+\rho^K\right) \\
&= \frac{-(K+1)\rho^{K+1}+K\rho^{K+2}+N\rho^N-(N-1)\rho^{N+1}}{(1-\rho)^2}.
\end{aligned} \tag{EC.62}
$$

The expression in (EC.61), which is equivalent to $L_p - NL_d$, satisfies the following relations:

$$\begin{aligned}
&\frac{\left(-\rho+(k+1)\rho^k-(k+1)\rho^{k+1}+\rho^{k+2}\right)N\rho\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})}\\
&+\frac{(N^N/N!)\left[-(N-1)\rho^{N+1}-(K+1)\rho^{K+1}-N(k+1)\rho^{N+k+1}+N(k+1)\rho^{N+k}+(N-1)\rho^{N+k+2}\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)^2(1-\rho^{k+1})}\\
&+\frac{(N^N/N!)\left[(K+N)\rho^{K+2}+(K+1-Nk-N)\rho^{K+k+2}+(Nk-K)\rho^{K+k+3}\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)^2(1-\rho^{k+1})}\\
&<\frac{\left(-\rho+(k+1)\rho^k\right)N\rho\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})}\\
&+\frac{\frac{N^N}{N!}\frac{1}{1-\rho}\left(-(N-1)\rho^{N+1}-(K+1)\rho^{K+1}+N(k+1)\rho^{N+k}+(N-1)\rho^{N+k+2}+(K+N)\rho^{K+2}\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})} \qquad \text{(EC.63)}\\
&<\frac{\left(-\rho+(k+1)\rho^k\right)N\rho\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})}\\
&+\frac{\frac{N^N}{N!}\frac{1}{1-\rho}\left(-(N-1)\rho^{N+1}+(N-1)\rho^{Nk+1}+N(k+1)\rho^{N+k}+(N-1)\rho^{N+k+2}\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})} \qquad \text{(EC.64)}\\
&<\frac{\left(-\rho+(k+1)\rho^k\right)N\rho\sum_{i=0}^{N-2}\frac{N^i}{i!}\rho^i}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})}+\frac{\frac{N^N}{N!}\frac{1}{1-\rho}\left(-(N-1)\rho^{N+1}+N(k+3)\rho^{N+k}\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K}\rho^i\right)(1-\rho)(1-\rho^{k+1})} \qquad \text{(EC.65)}\\
&<0. \qquad \text{(EC.66)}
\end{aligned}$$

The inequality (EC.66) completes the proof of Lemma EC.6. Below we will explain how we obtain each of the inequalities above.

The inequality (EC.63) holds because $-(k+1)\rho^{k+1}+\rho^{k+2}<-(k+1)\rho^{k+1}+\rho^{k+1}<0$ for $\rho<1$,

$$\begin{aligned}
&Nk\le K\le Nk+N-1, \text{ and}\\
&-N(k+1)\rho^{N+k+1}+(K+1-Nk-N)\rho^{K+k+2}+(Nk-K)\rho^{K+k+3}<0.
\end{aligned}$$

The inequality (EC.64) follows from the fact that $K\ge Nk$ and we have the following for $\rho<1$:

$$-(K+1)\rho^{K+1}+(K+N)\rho^{K+2}<-(K+1)\rho^{K+1}+(K+N)\rho^{K+1}=(N-1)\rho^{K+1}\le(N-1)\rho^{Nk+1}.$$

The inequality (EC.65) is because

$$\begin{aligned}
&-(N-1)\rho^{N+1}+(N-1)\rho^{Nk+1}+N(k+1)\rho^{N+k}+(N-1)\rho^{N+k+2}\\
&<-(N-1)\rho^{N+1}+(N-1)\rho^{N+k}+N(k+1)\rho^{N+k}+(N-1)\rho^{N+k}\\
&<-(N-1)\rho^{N+1}+N(k+3)\rho^{N+k}.
\end{aligned}$$

The inequality (EC.66) is due to the fact that $-\rho+(k+1)\rho^k<0$ and $-(N-1)\rho^{N+1}+N(k+3)\rho^{N+k}<0$ for $\rho<1$ and $k>z_3$. Below we will prove these two inequalities. We already know from the proof of Lemma EC.5 that

$g_4(z) \doteq (z+3)\rho^{z-1}$ is strictly decreasing in z for $\rho < 1$ and $z > -\frac{1}{\ln(\rho)} - 3$. Recall the definition of z_3 in (EC.55). Because $z_3 \geq -\frac{1}{\ln(\rho)} - 3$ and $(z_3+3)\rho^{z_3-1} \leq \frac{N-1}{N}$, we have the following for $k > z_3$:

$$(k+3)\rho^{k-1} < (N-1)/N.$$

Then,

$$-\rho + (k+1)\rho^k = \rho(-1+(k+1)\rho^{k-1}) < \rho\left(-1+(k+1)\frac{N-1}{N(k+3)}\right) < 0,$$

$$-(N-1)\rho^{N+1} + N(k+3)\rho^{N+k} = (N-1)\rho^{N+1}\left(-1+\frac{N}{N-1}(k+3)\rho^{k-1}\right) < 0. \tag{EC.67}$$

This completes the proofs of the claims that $-\rho+(k+1)\rho^k < 0$ and $-(N-1)\rho^{N+1}+N(k+3)\rho^{N+k} < 0$ for $\rho < 1$ and $k > z_3$. □

We now use the result in Lemma EC.6 to prove Theorem 1-(b)-(i) under the condition (15). For $\rho < 1$ and $k > z_3$,

$$W_p - W_d = \frac{L_p}{\lambda_{e,p}} - \frac{L_d}{\lambda_{e,d}} \tag{EC.68}$$

$$< \frac{NL_d}{\lambda_{e,p}} - \frac{L_d}{\lambda_{e,d}} \tag{EC.69}$$

$$< \frac{NL_d}{N\lambda_{e,d}} - \frac{L_d}{\lambda_{e,d}} \tag{EC.70}$$

$$= 0. \tag{EC.71}$$

The inequality (EC.68) follows from Little's Law. The inequality (EC.69) holds because $L_p < NL_d$ by Lemma EC.6. Recall from Proposition 1 that $\lambda_{e,p} > N\lambda_{e,d}$ regardless of the value of ρ. This implies the inequality (EC.70). The definition of k and (EC.71) complete the proof of Theorem 1-(b)-(i) under the condition (15).

We now show Theorem 1-(b)-(ii) under the condition (15). Recall that

$$SW_d = \lambda_{e,d}N(R - cW_d) \qquad \text{and} \qquad SW_p = \lambda_{e,p}(R - cW_p).$$

Because $\lambda_{e,d}N < \lambda_{e,p}$ by Proposition 1 and $W_d > W_p$ by part (b)-(i), we have $SW_d < SW_p$. This completes the proof of Theorem 1-(b)-(ii) under the condition (15). □

Appendix G: Statement and Proof of Lemma EC.7

LEMMA EC.7. *Consider any fixed service rate μ.* **(a)** *As $\rho \to \infty$, W_d and W_p satisfy the following relations:*

$$\lim_{\rho\to\infty} W_d(\rho) = \frac{\lfloor R\mu/c \rfloor}{\mu} \leq \lim_{\rho\to\infty} W_p(\rho) = \frac{\lfloor RN\mu/c \rfloor}{N\mu}, \tag{EC.72}$$

$$\lim_{\rho\to\infty} W_d'(\rho) = \lim_{\rho\to\infty} W_p'(\rho) = 0. \tag{EC.73}$$

(b) *As $\rho \to 1$, W_d and W_p satisfy the following relations:*

$$\lim_{\rho\to 1} [W_p(\rho) - W_d(\rho)] > 0 \tag{EC.74}$$

if $\lfloor RN\mu/c \rfloor - N\lfloor R\mu/c \rfloor > \left(\sum_{i=0}^{N-2} N^i/i!\right) / (N^N/N!)$, and for $R/c > 10/\mu$,

$$\lim_{\rho\to 1} \left[W_p'(\rho) - W_d'(\rho)\right] > (\lfloor R\mu/c \rfloor)^2 \gamma(N)/\mu^2 > 0, \tag{EC.75}$$

where $\gamma(N) > 0$ is a linear function of N and does not depend on other parameters.

(c) *As $\rho \to 0$, W_d and W_p satisfy the following relations:*

$$\lim_{\rho\to 0} W_d(\rho) = \lim_{\rho\to 0} W_p(\rho) = \frac{1}{\mu} \quad \text{and} \quad \lim_{\rho\to 0} W_d'(\rho) \geq \lim_{\rho\to 0} W_p'(\rho) = 0. \tag{EC.76}$$

Note that for a fixed μ, the limits $\rho \to 0$, $\rho \to 1$ and $\rho \to \infty$ are equivalent to $\lambda \to 0$, $\lambda \to \mu$ and $\lambda \to \infty$, respectively. In this proof, we will use these equivalent limits.

We first identify $W_d'(\lambda)$ and $W_p'(\lambda)$ and we will use those expressions to prove parts (a) through (c) of Proposition EC.7. Recall from (EC.7) that the average sojourn time in the dedicated system is

$$W_d(\lambda) = \frac{\rho-(k+1)\rho^{k+1}+k\rho^{k+2}}{\lambda(\rho^k-1)(\rho-1)} = \frac{1}{\mu}\frac{(1-(k+1)\rho^k+k\rho^{k+1})}{(\rho^k-1)(\rho-1)}, \quad \rho \neq 1. \tag{EC.77}$$

Thus, we have

$$\begin{aligned} W_d'(\lambda) =& \frac{1}{\mu^2}\frac{(-(k+1)k\rho^{k-1}+k(k+1)\rho^k)(\rho^k-1)(\rho-1)-(1-(k+1)\rho^k+k\rho^{k+1})((k+1)\rho^k-k\rho^{k-1}-1)}{(\rho-1)^2(\rho^k-1)^2} \\ =& \frac{1}{\mu^2}\frac{\rho^{2k}-k^2\rho^{k+1}+(2k^2-2)\rho^k-k^2\rho^{k-1}+1}{(\rho-1)^2(\rho^k-1)^2}. \end{aligned} \tag{EC.78}$$

Recall (8). When $K = N$,

$$W_p(\lambda) = \frac{1}{\mu}, \tag{EC.79}$$

and when $K > N$,

$$W_p(\lambda) = \frac{\sum_{i=0}^{N-1}\frac{N^i}{i!}i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K}i\rho^i}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)N\lambda} \tag{EC.80}$$

$$= \frac{\sum_{i=0}^{N}\frac{N^i}{i!}i\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K}i\rho^i}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)N\lambda}$$

$$= \frac{1}{(N\mu)\rho}\frac{N\rho\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K}i\rho^i}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i} \tag{EC.81}$$

$$= \frac{1}{N\mu}\frac{N\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K}i\rho^{i-1}}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i}. \tag{EC.82}$$

The equation (EC.81) is due to the fact that

$$\sum_{i=0}^{N}\frac{N^i}{i!}i\rho^i = \sum_{i=1}^{N}\frac{N^i}{(i-1)!}\rho^i = N\rho\sum_{i=1}^{N}\frac{N^{i-1}}{(i-1)!}\rho^{i-1} = N\rho\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i.$$

Based on (EC.79) and (EC.82),

$$W_p'(\lambda) = 0 \tag{EC.83}$$

for $K = N$, and when $K > N$, we have

$$\begin{aligned} W_p'(\lambda) =& \left[\frac{\partial}{\partial\rho}\left(\frac{N\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K}i\rho^{i-1}}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i}\right)\right]\frac{1}{N\mu^2} \\ =& \frac{\left(N\sum_{i=1}^{N-1}\frac{N^i}{i!}i\rho^{i-1} + \frac{N^N}{N!}\sum_{i=N+1}^{K}i(i-1)\rho^{i-2}\right)\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2 N\mu^2} \\ &- \frac{\left(N\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K}i\rho^{i-1}\right)\left(\sum_{i=1}^{N-1}\frac{N^i}{i!}i\rho^{i-1} + \frac{N^N}{N!}\sum_{i=N}^{K-1}i\rho^{i-1}\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2 N\mu^2}. \end{aligned} \tag{EC.84}$$

Proof of Part (a): From (EC.77), we have

$$\lim_{\lambda\to\infty} W_d(\lambda) = \lim_{\lambda\to\infty} \frac{1}{\mu} \frac{(1-(k+1)\rho^k + k\rho^{k+1})}{(\rho^k-1)(\rho-1)} = \frac{k}{\mu} \tag{EC.85}$$

because the leading term in the above numerator is $k\rho^{k+1}$ and the leading term in the above denominator is $\mu\rho^{k+1}$ as $\lambda \to \infty$.

From (EC.79), when $K = N$,

$$\lim_{\lambda\to\infty} W_p(\lambda) = \frac{1}{\mu} = \frac{K}{N\mu}. \tag{EC.86}$$

From (EC.82), it follows that for $K > N$,

$$\begin{aligned}\lim_{\lambda\to\infty} W_p(\lambda) =& \frac{1}{N\mu} \lim_{\lambda\to\infty} \frac{N\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K} i\rho^{i-1}}{\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1} \rho^i}\\ =& \frac{1}{N\mu} \lim_{\lambda\to\infty} \frac{N\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K} i\rho^{i-1}}{\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1} \rho^i}\\ =& \frac{K}{N\mu}.\end{aligned} \tag{EC.87}$$

Combining (EC.85) (EC.86), and (EC.87), and using the definitions of k and K, we get the relation in (EC.72).

To prove (EC.73), recall (EC.78), (EC.83) and (EC.84). Then,

$$\lim_{\lambda\to\infty} W_d'(\lambda) = \frac{1}{\mu^2} \lim_{\lambda\to\infty} \frac{\rho^{2k} - k^2\rho^{k+1} + (2k^2-2)\rho^k - k^2\rho^{k-1} + 1}{(\rho-1)^2(\rho^k-1)^2} = 0.$$

Furthermore, when $K = N$,

$$\lim_{\lambda\to\infty} W_p'(\lambda) = 0$$

and for $K > N$,

$$\begin{aligned}\lim_{\lambda\to\infty} W_p'(\lambda) =& \lim_{\lambda\to\infty} \frac{\left(N\sum_{i=1}^{N-1} \frac{N^i}{i!} i\rho^{i-1} + \frac{N^N}{N!}\sum_{i=N+1}^{K} i(i-1)\rho^{i-2}\right)\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2 N\mu^2}\\ &- \lim_{\lambda\to\infty} \frac{\left(N\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K} i\rho^{i-1}\right)\left(\sum_{i=1}^{N-1} \frac{N^i}{i!} i\rho^{i-1} + \frac{N^N}{N!}\sum_{i=N}^{K-1} i\rho^{i-1}\right)}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2 N\mu^2}\\ =& \frac{1}{N\mu^2}\lim_{\lambda\to\infty} \frac{\left(N\sum_{i=1}^{N-1} \frac{N^i}{i!} i\rho^{i-1} + \frac{N^N}{N!}\sum_{i=N+1}^{K} i(i-1)\rho^{i-2}\right)\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2}\\ &- \frac{1}{N\mu^2}\lim_{\lambda\to\infty} \frac{\left(N\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K} i\rho^{i-1}\right)\left(\sum_{i=1}^{N-1} \frac{N^i}{i!} i\rho^{i-1} + \frac{N^N}{N!}\sum_{i=N}^{K-1} i\rho^{i-1}\right)}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2}\\ =&0.\end{aligned}$$

This completes the proof of (EC.73). □

Proof of Part (b): Note from (EC.77) that $W_d(\lambda)$ can also be expressed as the following:

$$W_d(\lambda) = \frac{\sum_{i=0}^{k} i\rho^i}{\lambda\sum_{i=0}^{k-1}\rho^i} = \frac{\sum_{i=0}^{k} i\rho^i}{\mu\rho\sum_{i=0}^{k-1}\rho^i}. \tag{EC.88}$$

Then, we have

$$\lim_{\lambda\to\mu} W_d(\lambda) = \frac{1}{\mu}\left(\lim_{\lambda\to\mu}\frac{\sum_{i=0}^{k} i\rho^i}{\rho\sum_{i=0}^{k-1}\rho^i}\right) = \frac{k+1}{2\mu}. \tag{EC.89}$$

In addition, observe from (EC.82) that when $K > N$,

$$\begin{aligned}\lim_{\lambda\to\mu} W_p(\lambda) = \frac{1}{N\mu}\lim_{\lambda\to\mu}\frac{N\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K} i\rho^{i-1}}{\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i} &= \frac{N\sum_{i=0}^{N-1}\frac{N^i}{i!} + \frac{N^N}{N!}\sum_{i=N+1}^{K} i}{\sum_{i=0}^{N-1}\frac{N^i}{i!} + \frac{N^N}{N!}(K-N)}\left(\frac{1}{N\mu}\right)\\ &= \frac{N\sum_{i=0}^{N-1}\frac{N^i}{i!} + \frac{N^N}{N!}\frac{(K+N+1)(K-N)}{2}}{\sum_{i=0}^{N-1}\frac{N^i}{i!} + \frac{N^N}{N!}(K-N)}\left(\frac{1}{N\mu}\right)\\ &= \left(N + \frac{1}{2}\frac{\frac{N^N}{N!}(K-N+1)(K-N)}{\sum_{i=0}^{N-2}\frac{N^i}{i!} + \frac{N^N}{N!}(K-N+1)}\right)\left(\frac{1}{N\mu}\right).\end{aligned} \tag{EC.90}$$

The equation (EC.90) also holds when $K = N$ since $W_p(\lambda) = \frac{1}{\mu}$ when $K = N$.

By (EC.90) and the definitions of k and K from (5) and (7), we have

$$\begin{aligned}\lim_{\lambda\to\mu} W_p(\lambda) &= \left(N + \frac{1}{2}\frac{\frac{N^N}{N!}(K-N+1)(K-N)}{\sum_{i=0}^{N-2}\frac{N^i}{i!} + \frac{N^N}{N!}(K-N+1)}\right)\frac{1}{N\mu}\\ &= \left(N + \frac{K-N}{2} - \frac{K-N}{2} + \frac{1}{2}\frac{\frac{N^N}{N!}(K-N+1)(K-N)}{\sum_{i=0}^{N-2}\frac{N^i}{i!} + \frac{N^N}{N!}(K-N+1)}\right)\frac{1}{N\mu}\\ &= \left(\frac{K+N}{2} - \frac{(K-N)\sum_{i=0}^{N-2}\frac{N^i}{i!}}{2(\sum_{i=0}^{N-2}\frac{N^i}{i!} + \frac{N^N}{N!}(K-N+1))}\right)\frac{1}{N\mu}\\ &> \left(\frac{K+N}{2} - \frac{(K-N)\sum_{i=0}^{N-2}\frac{N^i}{i!}}{2(\frac{N^N}{N!}(K-N+1))}\right)\frac{1}{N\mu}\\ &> \left(K+N - \frac{\sum_{i=0}^{N-2}\frac{N^i}{i!}}{\frac{N^N}{N!}}\right)\frac{1}{2N\mu}.\end{aligned} \tag{EC.91}$$

We already know from (EC.89) that $\lim_{\lambda\to\mu} W_d(\lambda) = \frac{k+1}{2\mu}$. This and (EC.91) imply that

$$\begin{aligned}\lim_{\lambda\to\mu}[W_p(\lambda) - W_d(\lambda)] &> \left(K+N - \frac{\sum_{i=0}^{N-2}\frac{N^i}{i!}}{\frac{N^N}{N!}}\right)\frac{1}{2N\mu} - \frac{k+1}{2\mu}\\ &= \left(K - Nk - \frac{\sum_{i=0}^{N-2}\frac{N^i}{i!}}{\frac{N^N}{N!}}\right)\frac{1}{2N\mu}.\end{aligned}$$

Therefore, if $K - Nk > \frac{\sum_{i=0}^{N-2}\frac{N^i}{i!}}{\frac{N^N}{N!}}$, $\lim_{\lambda\to\mu}[W_p(\lambda) - W_d(\lambda)] > 0$.

We now show (EC.75). Suppose that $R/c > 10/\mu$. Then, $k \geq 10$ and $K > N$. Recall (EC.88). Then,

$$\begin{aligned}\lim_{\lambda\to\mu} W_d'(\lambda) = \frac{1}{\mu^2}\lim_{\lambda\to\mu}\frac{\partial}{\partial\rho}\left(\frac{\sum_{i=0}^{k} i\rho^i}{\rho\sum_{i=0}^{k-1}\rho^i}\right) &= \frac{1}{\mu^2}\lim_{\lambda\to\mu}\frac{\left(\sum_{i=1}^{k} i^2\rho^{i-1}\right)\left(\sum_{i=1}^{k}\rho^i\right) - \left(\sum_{i=0}^{k} i\rho^i\right)\left(\sum_{i=1}^{k} i\rho^{i-1}\right)}{(\sum_{i=1}^{k}\rho^i)^2}\\ &= \frac{1}{\mu^2}\frac{\frac{k(k+1)(2k+1)}{6}k - \frac{k(k+1)}{2}\frac{k(k+1)}{2}}{k^2}\\ &= \frac{k^2-1}{12\mu^2}.\end{aligned} \tag{EC.92}$$

Recalling (EC.84), we now find $\lim_{\lambda\to\mu} W_p'(\lambda)$:

$$\begin{aligned}
&\lim_{\lambda\to\mu} W_p'(\lambda)\\
&=\frac{1}{N\mu^2}\lim_{\lambda\to\mu}\frac{\left(N\sum_{i=1}^{N-1}\frac{N^i}{i!}i\rho^{i-1}+\frac{N^N}{N!}\sum_{i=N+1}^{K}i(i-1)\rho^{i-2}\right)\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2}\\
&-\frac{1}{N\mu^2}\lim_{\lambda\to\mu}\frac{\left(N\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N+1}^{K}i\rho^{i-1}\right)\left(\sum_{i=1}^{N-1}\frac{N^i}{i!}i\rho^{i-1}+\frac{N^N}{N!}\sum_{i=N}^{K-1}i\rho^{i-1}\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}\rho^i+\frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2}\\
&=\frac{1}{N\mu^2}\frac{\left(N\sum_{i=1}^{N-1}\frac{N^i}{i!}i+\frac{N^N}{N!}\sum_{i=N+1}^{K}i(i-1)\right)\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2}\\
&-\frac{1}{N\mu^2}\frac{\left(N\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}\sum_{i=N+1}^{K}i\right)\left(\sum_{i=1}^{N-1}\frac{N^i}{i!}i+\frac{N^N}{N!}\sum_{i=N}^{K-1}i\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2}\\
&>\frac{\left(\frac{N^N}{N!}\sum_{i=N+1}^{K}i(i-1)\right)\left(\frac{N^N}{N!}(K-N)\right)-\left(N\sum_{i=0}^{N-1}\frac{N^N}{N!}+\frac{N^N}{N!}\sum_{i=N+1}^{K}i\right)\left(\sum_{i=1}^{N-1}\frac{N^N}{N!}i+\frac{N^N}{N!}\sum_{i=N}^{K-1}i\right)}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2N\mu^2}
\end{aligned}\tag{EC.93}$$

$$\begin{aligned}
&=\frac{\left(\frac{N^N}{N!}\right)^2\left[\left(\sum_{i=N+1}^{K}i(i-1)(K-N)\right)-\left(\frac{K^2+N^2+K-N}{2}\right)\left(\frac{K(K-1)}{2}\right)\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2N\mu^2}\\
&=\frac{\left(\frac{N^N}{N!}\right)^2\left[\left(\frac{K(K+1)(2K+1)-N(N+1)(2N+1)}{6}-\frac{(K+N+1)(K-N)}{2}\right)(K-N)-\left(\frac{K^2+K+N^2-N}{2}\right)\left(\frac{K(K-1)}{2}\right)\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2N\mu^2}\\
&>\frac{\left(\frac{N^N}{N!}\right)^2\left[\left(\frac{K(K+1)(2K+1)-N(N+1)(2N+1)}{6}-\frac{K^2+K}{2}\right)(K-N)-\left(\frac{K^2+K+N^2-N}{2}\right)\left(\frac{K(K-1)}{2}\right)\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2N\mu^2}.
\end{aligned}\tag{EC.94}$$

The inequality (EC.93) is because $\frac{N^i}{i!}<\frac{N^N}{N!}$ for $i\in\{1,2,\ldots,N-2\}$ and $\frac{N^{N-1}}{(N-1)!}=\frac{N^N}{N!}$. Note that the condition $R/c>10/\mu$ in the statement of part (b) is equivalent to $k\geq 10$. Then, because $K\geq Nk$, $k\geq 10$ implies $N\leq\frac{K}{10}$. Using this and (EC.94), we have

$$\begin{aligned}
&\lim_{\lambda\to\mu} W_p'(\lambda)\\
&>\frac{\left(\frac{N^N}{N!}\right)^2\left[\left(\frac{K(K+1)(2K+1)-N(N+1)(2N+1)}{6}-\frac{K^2+K}{2}\right)(K-N)-\left(\frac{K^2+K+N^2-N}{2}\right)\left(\frac{K(K-1)}{2}\right)\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2N\mu^2}\\
&\geq\frac{\left(\frac{N^N}{N!}\right)^2\left[\left(\frac{K(K+1)(2K+1)-\frac{K}{10}\left(\frac{K}{10}+1\right)\left(\frac{2K}{10}+1\right)}{6}-\frac{K^2+K}{2}\right)(K-N)-\left(\frac{K^2+K+(\frac{K}{10})^2-\frac{K}{10}}{2}\right)\left(\frac{K(K-1)}{2}\right)\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2N\mu^2}
\end{aligned}\tag{EC.95}$$

$$\geq\frac{\left(\frac{N^N}{N!}\right)^2\left[\left(\frac{K(K+1)(2K+1)-\frac{K}{10}\left(\frac{K}{10}+1\right)\left(\frac{2K}{10}+1\right)}{6}-\frac{K^2+K}{2}\right)\left(K-\frac{K}{10}\right)-\left(\frac{K^2+K+(\frac{K}{10})^2-\frac{K}{10}}{2}\right)\left(\frac{K(K-1)}{2}\right)\right]}{\left(\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}(K-N)\right)^2N\mu^2}\tag{EC.96}$$

$$= \frac{\left(\frac{N^N}{N!}\right)^2 (236K^2 + 115K - 450)K^2}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} + \frac{N^N}{N!}(K-N)\right)^2 5000N\mu^2}$$

$$> \frac{\left(\frac{N^N}{N!}\right)^2 236N^2k^2K^2}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} + \frac{N^N}{N!}(K-N)\right)^2 5000N\mu^2} \tag{EC.97}$$

$$> \frac{\left(\frac{N^N}{N!}\right)^2 K^2}{\left(\sum_{i=0}^{N-1} \frac{N^N}{N!} + \frac{N^N}{N!}(K-N)\right)^2 N\mu^2} \frac{236N^2k^2}{5000}$$

$$= \frac{236Nk^2}{5000\mu^2} \tag{EC.98}$$

$$> \frac{k^2-1}{12\mu^2} = \lim_{\lambda\to\mu} W_d'(\lambda). \tag{EC.99}$$

The inequality (EC.95) is because $K - N > 0$, $\frac{K(K-1)}{2} > 0$, and $N \le \frac{K}{10}$. The inequality (EC.96) holds because $N \le \frac{K}{10}$ and $\frac{K(K+1)(2K+1) - \frac{K}{10}\left(\frac{K}{10}+1\right)\left(\frac{2K}{10}+1\right)}{6} - \frac{K^2+K}{2} > 0$ as $K \ge Nk \ge 20$. The inequality (EC.97) follows from the fact that $K \ge Nk$ and $115K - 450 > 0$ because $K \ge Nk \ge 20$ for $N \ge 2$ and $k \ge 10$. The inequality (EC.99) follows from the fact that $N \ge 2$. Thus, $\lim_{\lambda\to\mu} W_p'(\lambda) > \lim_{\lambda\to\mu} W_d'(\lambda)$ for $k \ge 10$. Based on the inequalities above we now show (EC.75). Using (EC.92) and (EC.98), we have

$$\lim_{\lambda\to\mu} \left[W_p'(\lambda) - W_d'(\lambda)\right] > \frac{236Nk^2}{5000\mu^2} - \frac{k^2-1}{12\mu^2} > \frac{k^2}{\mu^2}\gamma(N) > 0, \tag{EC.100}$$

where $\gamma(N) \doteq \left(\frac{236N}{5000} - \frac{1}{12}\right)$. This completes the proof of our claim in (EC.75). □

Proof of Part (c): Recall (EC.77), (EC.78), (EC.79), (EC.82), (EC.83) and (EC.84). Then, we have

$$\lim_{\lambda\to 0} W_d(\lambda) = \frac{1}{\mu} \lim_{\lambda\to 0} \frac{(1-(k+1)\rho^k + k\rho^{k+1})}{(\rho^k-1)(\rho-1)} = \frac{1}{\mu}.$$

If $k > 1$, we get

$$\lim_{\lambda\to 0} W_d'(\lambda) = \frac{1}{\mu^2} \lim_{\lambda\to 0} \frac{\rho^{2k} - k^2\rho^{k+1} + (2k^2-2)\rho^k - k^2\rho^{k-1} + 1}{(\rho-1)^2(\rho^k-1)^2} = \frac{1}{\mu^2};$$

otherwise, that is, if $k = 1$, $\lim_{\lambda\to 0} W_d'(\lambda) = 0$. In addition, for $K = N$,

$$\lim_{\lambda\to 0} W_p(\lambda) = \frac{1}{\mu} \text{ and } \lim_{\lambda\to 0} W_p'(\lambda) = 0.$$

and for $K > N$,

$$\lim_{\lambda\to 0} W_p(\lambda) = \frac{1}{N\mu} \lim_{\lambda\to 0} \frac{N\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K} i\rho^{i-1}}{\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1} \rho^i} = \frac{1}{N\mu}N = \frac{1}{\mu},$$

and

$$\lim_{\lambda\to 0} W_p'(\lambda) = \frac{1}{N\mu^2} \lim_{\lambda\to 0} \frac{\left(N\sum_{i=1}^{N-1} \frac{N^i}{i!} i\rho^{i-1} + \frac{N^N}{N!}\sum_{i=N+1}^{K} i(i-1)\rho^{i-2}\right)\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2}$$

$$- \frac{1}{N\mu^2} \lim_{\lambda\to 0} \frac{\left(N\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N+1}^{K} i\rho^{i-1}\right)\left(\sum_{i=1}^{N-1} \frac{N^i}{i!} i\rho^{i-1} + \frac{N^N}{N!}\sum_{i=N}^{K-1} i\rho^{i-1}\right)}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \frac{N^N}{N!}\sum_{i=N}^{K-1}\rho^i\right)^2} \tag{EC.101}$$

$$= \frac{1}{N\mu^2}(N^2 - N^2) = 0. \tag{EC.102}$$

This completes the proof of part (c). □

Appendix H: Proof of Theorem 2

Recall Remark 2. In this section, we will prove a generalized version of Theorem 2, i.e., Theorem EC.1, which is valid both when ν is an integer and when ν is not an integer. We will first state Theorem EC.1, and then provide its proof. Theorem 2 is an immediate corollary of Theorem EC.1.

To state Theorem EC.1, we shall introduce new notation. Let m be the minimum system size n that makes $Rn\mu/c$ an integer:

$$m \doteq \min\left\{n \in \mathbb{N}_+ : \frac{Rn\mu}{c} \in \mathbb{N}_+\right\}. \tag{EC.103}$$

For instance, in Figure 3, $m = 5$. Because of this, we observe a repeating pattern (in which percentages increase with N) for every fifth consecutive data point starting from any single point on Figure 3. We now define a subsequence that considers every i^{th} system size in each repeating pattern. Specifically, for each $i = 1, 2, \ldots, m$, define a system size subsequence $\mathcal{S}_i \doteq \{N_{i,0}, N_{i,1}, N_{i,2}, N_{i,3}, \ldots\}$ such that

$$N_{i,\ell} = i + \ell m, \quad \ell = 0, 1, \ldots \tag{EC.104}$$

For example, in Figure 3, the system size subsequence $\mathcal{S}_3 = \{3, 8, 13, 18, \ldots\}$ includes the third system size ($i = 3$) in each repeating pattern that consists of 5 data points ($m = 5$).

THEOREM EC.1. *Let $i = 1, 2\ldots, m$. Then, we have the following results:*
(a) *There exists a constant η_1 such that the subsequence $\beta_W(N_{i,\cdot}) \doteq \left\{\frac{(W_p(N_{i,\ell}) - W_d(N_{i,\ell}))}{W_d(N_{i,\ell})},\ \ell = 0, 1, \ldots\right\}$ is non-negative and strictly increasing in the system size if $\rho > 1$ and $\nu > \eta_1$. The constant η_1 is finite when $\rho > 1$, and does not depend on either R or c.*
(b) *There exists a constant η_2 such that the subsequence $\beta_{SW}(N_{i,\cdot}) \doteq \left\{\frac{(SW_d(N_{i,\ell}) - SW_p(N_{i,\ell}))}{SW_p(N_{i,\ell})},\ \ell = 0, 1, \ldots\right\}$ is non-negative and strictly increasing in the system size if $\rho > 1$ and $\nu > \eta_2$. The constant η_2 is finite when $\rho > 1$, and does not depend on either R or c.*

Let us define some constants which will be used in the remainder of the proof:

$$\eta_1 \doteq \max\{z_0, z_4, z_5\} + 1 \quad \text{and} \quad \eta_2 \doteq \max\{z_0, z_6, z_7, z_8\} + 1, \tag{EC.105}$$

where

$$z_0 \doteq \inf\left\{z \in \mathbb{R} : \frac{2(z+1)\rho}{\rho^{z+1} - 1} < \frac{1}{4(\rho-1)^2}, \frac{z+1}{\rho^z} < \frac{1}{4} \text{ and } z > \frac{1}{\ln(\rho)} - 1\right\}, \tag{EC.106}$$

$$z_4 \doteq \inf\left\{z \in \mathbb{R} : \frac{(\rho^z - 1)^2}{z^2\rho^z} > 2(\rho-1)\ln(\rho) \text{ and } z > \frac{2}{\ln(\rho)}\right\}, \tag{EC.107}$$

$$z_5 \doteq \inf\left\{z \in \mathbb{R} : \frac{2(z+1)}{\rho^{z-1} - 1} < \frac{\rho}{2(\rho-1)^2} \text{ and } z > \frac{3}{\ln(\rho)} + 1\right\}, \tag{EC.108}$$

$$z_6 \doteq \inf\left\{z \in \mathbb{R} : \frac{z}{(\rho^z - 1)} < \frac{3}{8\rho^2}, z > \frac{1}{\ln(\rho)}\right\}, \tag{EC.109}$$

$$z_7 \doteq \inf\left\{z \in \mathbb{R} : \frac{(\rho^z - 1)}{z^3} > \ln(\rho)4\rho^2, z > \frac{3}{\ln(\rho)}\right\}, \tag{EC.110}$$

$$z_8 \doteq \inf\left\{z \in \mathbb{R} : 2(z+2)\frac{1}{\rho^{(z-2)} - 1} < \frac{1}{10(\rho-1)}, z > \frac{2}{\ln(\rho)} + 2\right\}. \tag{EC.111}$$

We now state and prove Lemma EC.8 that shows the existence of constants z_0 and z_4 through z_8, which are defined in (EC.106) through (EC.111), respectively.

LEMMA EC.8. *The following claims hold for $\rho > 1$:* **(a)***The constant z_4 defined in $(EC.107)$ exists and it is finite.* **(b)** *The constant z_5 defined in $(EC.108)$ exists and it is finite.* **(c)** *The constant z_6 defined in $(EC.109)$ exists and it is finite.* **(d)** *The constant z_7 defined in $(EC.110)$ exists and it is finite.* **(e)** *The constant z_8 defined in $(EC.111)$ exists and it is finite.* **(f)** *The constant z_0 defined in $(EC.106)$ exists and it is finite.*

Proof of Lemma EC.8: Part (a): Define $g_5(z) \doteq \frac{(\rho^z-1)^2}{z^2\rho^z}$. Then, note that the definition in (EC.107) is equivalent to $z_4 \doteq \inf\{z \in \mathbb{R} : g_5(z) > 2(\rho-1)\ln(\rho) \text{ and } z > 2/\ln(\rho)\}$. The function $g_5(\cdot)$ is strictly increasing when $\rho > 1$ and $z > \frac{2}{\ln(\rho)}$ because

$$\begin{aligned} g_5'(z) =& \frac{2(\rho^z-1)\rho^z\ln(\rho)z^2\rho^z - (\rho^z-1)^2(2z\rho^z + z^2\rho^z\ln(\rho))}{(z^2\rho^z)^2} \\ >& \frac{2(\rho^z-1)\rho^z\ln(\rho)z^2\rho^z - \rho^z(\rho^z-1)(2z\rho^z + z^2\rho^z\ln(\rho))}{(z^2\rho^z)^2} \\ =& \frac{(\rho^z-1)z\rho^{2z}(2\ln(\rho)z - (2+\ln(\rho)z))}{(z^2\rho^z)^2} \\ =& \frac{(\rho^z-1)(\ln(\rho)z-2)}{z^3} \\ >& 0 \end{aligned} \tag{EC.112}$$

for $\rho > 1$ and $z > \frac{2}{\ln(\rho)}$. In addition, when $\rho > 1$, we have

$$\lim_{z\to\infty} g_5(z) = \lim_{z\to\infty} \frac{(\rho^z-1)^2}{z^2\rho^z} = \lim_{z\to\infty} \frac{2(\rho^z-1)\rho^z\ln(\rho)}{2z\rho^z + z^2\rho^z\ln(\rho)} = \lim_{z\to\infty} \frac{2(\rho^z-1)\ln(\rho)}{2z + z^2\ln(\rho)} = \lim_{z\to\infty} \frac{2\rho^z(\ln(\rho))^2}{2+2z\ln(\rho)} = \infty. \tag{EC.113}$$

It follows from (EC.112) and (EC.113) that z_4 exists and it is finite. □ **Part (b):** Define $g_6(z) \doteq \frac{2(z+1)}{\rho^{z-1}-1}$. Then, the definition in (EC.108) is equivalent to $z_5 \doteq \inf\{z \in \mathbb{R} : g_6(z) < \frac{\rho}{2(\rho-1)^2} \text{ and } z > \frac{3}{\ln(\rho)} + 1\}$. The function $g_6(\cdot)$ is strictly decreasing when $\rho > 1$ and $z > \frac{3}{\ln(\rho)} + 1$ because

$$g_6'(z) = \frac{2(\rho^{z-1}-1) - 2(z+1)(\rho^{z-1})\ln(\rho)}{(\rho^{z-1}-1)^2} < \frac{2\rho^{z-1}(1-(z+1)\ln(\rho))}{(\rho^{z-1}-1)^2} < 0 \tag{EC.114}$$

for $\rho > 1$ and $z > \frac{3}{\ln(\rho)} + 1$. Furthermore, when $\rho > 1$, we have

$$\lim_{z\to\infty} g_6(z) = \lim_{z\to\infty} \frac{2(z+1)}{\rho^{z-1}-1} = \lim_{z\to\infty} \frac{2}{\rho^{z-1}\ln(\rho)} = 0. \tag{EC.115}$$

It follows from (EC.114) and (EC.115) that z_5 exists and it is finite. □ **Part (c):** Define $g_7(z) \doteq \frac{z}{(\rho^z-1)}$. Then, the definition in (EC.109) is equivalent to $z_6 = \inf\{z \in \mathbb{R} : g_7(z) < \frac{3}{8\rho^2}, z > \frac{1}{\ln(\rho)}\}$. The function $g_7(\cdot)$ is strictly decreasing when $\rho > 1$ and $z > \frac{1}{\ln(\rho)}$ because

$$g_7'(z) = \frac{\rho^z - 1 - z\rho^z\ln(\rho)}{(\rho^z-1)^2} < \frac{\rho^z(1-zln(\rho))}{(\rho^z-1)^2} < 0 \tag{EC.116}$$

for $\rho > 1$ and $z > \frac{1}{\ln(\rho)}$. Moreover, when $\rho > 1$, we have

$$\lim_{z\to\infty} g_7(z) = \lim_{z\to\infty} \frac{z}{(\rho^z-1)} = \lim_{z\to\infty} \frac{1}{\rho^z\ln(\rho)} = 0. \tag{EC.117}$$

It follows from (EC.116) and (EC.117) that z_6 exists and it is finite. □ **Part (d):** Define $g_8(z) \doteq \frac{(\rho^z-1)}{z^3}$. Then, the definition in (EC.110) is equivalent to $z_7 = \inf\{z \in \mathbb{R} : g_8(z) > \ln(\rho)4\rho^2, z > \frac{3}{\ln(\rho)}\}$. The function $g_8(\cdot)$ is strictly increasing when $\rho > 1$ and $z > \frac{3}{\ln(\rho)}$ because

$$g_8'(z) = \frac{\rho^z\ln(\rho)z^3 - (\rho^z-1)3z^2}{z^6} > \frac{(\ln(\rho)z-3)\rho^z}{z^4} > 0 \tag{EC.118}$$

for $\rho > 1$ and $z > \frac{3}{\ln(\rho)}$. Also, for $\rho > 1$, we have

$$\lim_{z\to\infty} g_8(z) = \lim_{z\to\infty} \frac{(\rho^z - 1)}{z^3} = \lim_{z\to\infty} \frac{\rho^z \ln(\rho)}{3z^2} = \lim_{z\to\infty} \frac{\rho^z (\ln(\rho))^2}{6z} = \lim_{z\to\infty} \frac{\rho^z (\ln(\rho))^3}{6} = \infty. \tag{EC.119}$$

By (EC.118) and (EC.119), z_7 exists and it is finite. □ **Part (e):** Define $g_9(z) \doteq 2(z+2)\frac{1}{\rho^{(z-2)}-1}$. Then, the definition in (EC.111) is equivalent to $z_8 = \inf\{z \in \mathbb{R} : g_9(z) < \frac{1}{10(\rho-1)}, z > \frac{2}{\ln(\rho)} + 2\}$. Note that $g_9(z)$ is strictly decreasing when $\rho > 1$ and $z > \frac{2}{\ln(\rho)} + 2$ because

$$g_9'(z) = 2\frac{(\rho^{(z-2)} - 1) - (z+2)\rho^{(z-2)}\ln(\rho)}{(\rho^{(z-2)} - 1)^2} < 2(\rho^{(z-2)} - 1)\frac{1-(z+2)\ln(\rho)}{(\rho^{(z-2)} - 1)^2} < 0 \tag{EC.120}$$

for $\rho > 1$ and $z > \frac{2}{\ln(\rho)} + 2$. Furthermore, for $\rho > 1$, we have

$$\lim_{z\to\infty} g_9(z) = \lim_{z\to\infty} 2(z+2)\frac{1}{\rho^{(z-2)} - 1} = \lim_{z\to\infty} \frac{2}{\rho^{(z-2)}\ln(\rho)} = 0. \tag{EC.121}$$

By (EC.120) and (EC.121), z_8 exists and it is finite. **Part (f):** Define $g_{10}(z) \doteq \frac{2(z+1)\rho}{\rho^{z+1}-1}$ and $g_{11}(z) \doteq \frac{z+1}{\rho^z}$. Then, the definition in (EC.106) is equivalent to $z_0 = \inf\{z \in \mathbb{R} : g_{10}(z) < \frac{1}{4(\rho-1)^2}, g_{11}(z) < \frac{1}{4}, z > \frac{1}{\ln(\rho)} - 1\}$. Note that both $g_{10}(z)$ and $g_{11}(z)$ are strictly decreasing when $\rho > 1$ and $z > \frac{1}{\ln(\rho)} - 1$ because

$$g_{10}'(z) = 2\rho\frac{(\rho^{z+1} - 1) - (z+1)(\rho^{z+1}\ln(\rho))}{(\rho^{z+1} - 1)^2} = 2\rho\frac{\rho^{z+1}(1-(z+1)\ln(\rho)) - 1}{(\rho^{z+1} - 1)^2} < 0 \text{ and} \tag{EC.122}$$

$$g_{11}'(z) = \frac{1-(z+1)\ln(\rho)}{\rho^z} < 0 \tag{EC.123}$$

for $\rho > 1$ and $z > \frac{1}{\ln(\rho)} - 1$. In addition, for $\rho > 1$, we have

$$\lim_{z\to\infty} g_{10}(z) = \lim_{z\to\infty} \frac{2(z+1)\rho}{\rho^{(z+1)} - 1} = \lim_{z\to\infty} \frac{2\rho}{\rho^{(z+1)}\ln(\rho)} = 0, \tag{EC.124}$$

$$\lim_{z\to\infty} g_{11}(z) = \lim_{z\to\infty} \frac{z+1}{\rho^z} = \lim_{z\to\infty} \frac{1}{\rho^z \ln(\rho)} = 0. \tag{EC.125}$$

By (EC.122) through (EC.125), z_0 exists and it is finite. □

Proof of Theorem 2 - Part (a): Take any $i \in \{1, 2, \dots, m\}$. Recall the definition of η_1 from (EC.105). First, we will show that the percentage subsequence $\beta_W(N_{i,\cdot})$ is non-negative under the stated conditions in part (a). When $N_{i,\ell} = 1$, $\frac{W_p(N_{i,\ell}) - W_d(N_{i,\ell})}{W_p(N_{i,\ell})} = 0$. Recall the definition of z_1 from (EC.24). Below we will show by (EC.126) through (EC.128) that $z_0 \geq z_1(N_{i,\ell})$ for any $N_{i,\ell} \geq 2$ when $\rho > 1$. Thus, $k > z_0$ implies $k > z_1(N_{i,\ell})$ for $\rho > 1$. Then, it follows from Theorem 1-(a)-(i) that $W_p(N_{i,\ell}) - W_d(N_{i,\ell}) > 0$ for any $N_{i,\ell} \geq 2$ and hence $\frac{W_p(N_{i,\ell}) - W_d(N_{i,\ell})}{W_p(N_{i,\ell})} > 0$ when $\rho > 1$ and $k > z_0$. This and the fact that $R/c > (z_0 + 1)/\mu$ implies $k > z_0$ complete our argument for the statement that $\beta_W(N_{i,\cdot})$ subsequence is non-negative for any i under the stated conditions in part (a). We now show our above claim that $z_0 \geq z_1(N_{i,\ell})$ for any $N_{i,\ell} \geq 2$ when $\rho > 1$. To do so, for any $N_{i,\ell} \geq 2$, we will first show that z_0 already meets the conditions $z_1(N_{i,\ell})$ meets.

$$\begin{aligned}
\frac{(z_0+1)N_{i,\ell}\rho}{\rho^{z_0+1} - 1} - \frac{N_{i,\ell} - 1}{4(\rho-1)^2} &= (N_{i,\ell} - 1)\left(\frac{N_{i,\ell}}{N_{i,\ell} - 1}\frac{(z_0+1)\rho}{\rho^{z_0+1} - 1} - \frac{1}{4(\rho-1)^2}\right) \\
&\leq (N_{i,\ell} - 1)\left(2\frac{(z_0+1)\rho}{\rho^{z_0+1} - 1} - \frac{1}{4(\rho-1)^2}\right) \\
&\leq 0,
\end{aligned} \tag{EC.126}$$

$$\frac{(z_0+1)N_{i,\ell}}{(N_{i,\ell} - 1)\rho^{z_0}} - \frac{1}{2} \leq \frac{2(z_0+1)}{\rho^{z_0}} - \frac{1}{2} \leq 0, \tag{EC.127}$$

$$z_0 > \frac{1}{\ln(\rho)} - 1. \tag{EC.128}$$

Based on these, suppose for a contradiction that $z_0 < z_1(N_{i,\ell})$ for some $N_{i,\ell} \geq 2$. Then, for any $z \in (z_0, z_1(N_{i,\ell}))$, z satisfies the set of constraints that defines $z_1(N_{i,\ell})$ since $g_1(z)$ and $g_2(z)$ (defined in the proof of Proposition 3) are strictly decreasing in z when $\rho > 1$ and $z > 1/\ln(\rho) - 1$. But, this contradicts with the definition of $z_1(N_{i,\ell})$. Thus, $z_0 \geq z_1(N_{i,\ell})$.

Next, we prove that the percentage subsequence is strictly increasing when $\rho > 1$ and $k > \max\{z_4, z_5\}$. Take any $\ell_1 \in \mathbb{N}$ and $\ell_2 \in \mathbb{N}_+$ such that $\ell_1 < \ell_2$. Let $N_1 \doteq i + \ell_1 m$ and $N_2 \doteq i + \ell_2 m$, which imply that $N_1 < N_2$ and $\{N_1, N_2\} \subset \{N_{i,\ell} : \ell = 0, 1, \ldots\}$.Based on these, the outline of the remainder of our proof is as follows. By Proposition 2, $W_p(N_2) > W_s(N_2)$. Thus,

$$W_p(N_2) - W_p(N_1) > W_s(N_2) - W_p(N_1) \tag{EC.129}$$

$$= W_s(N_2) - W_s(N_1) - (W_p(N_1) - W_s(N_1)). \tag{EC.130}$$

We claim and show below that if $\rho > 1$ and $k > \max\{z_4, z_5\}$,

$$W_s(N_2) - W_s(N_1) > \frac{\rho - 1}{\lambda N_1(N_1+1)} \frac{\rho}{2(\rho-1)^2}, \tag{EC.131}$$

and

$$W_p(N_1) - W_s(N_1) < \frac{\rho - 1}{\lambda N_1(N_1+1)} \frac{\rho}{2(\rho-1)^2}. \tag{EC.132}$$

Combining (EC.130) through (EC.132), we have

$$W_p(N_2) - W_p(N_1) > 0 \tag{EC.133}$$

if $\rho > 1$ and $k > \max\{z_4, z_5\}$. Recall from (EC.7) that

$$W_d(N_1) = W_d(N_2) = \frac{1}{\lambda(\rho-1)} \frac{k\rho^{k+2} - (k+1)\rho^{k+1} + \rho}{\rho^k - 1}.$$

Since $W_d(N)$ does not depend on N, by (EC.133), $W_p(N_{i,\ell}) - W_d(N_{i,\ell})$ is strictly increasing in ℓ when $\rho > 1$ and $k > \max\{z_4, z_5\}$. As a result, $\beta_W(N_{i,\cdot}) = \frac{(W_p(N_{i,\cdot}) - W_d(N_{i,\cdot}))}{W_d(N_{i,\cdot})}$ is also strictly increasing in system size (i.e., ℓ) when $\rho > 1$ and $k > \max\{z_4, z_5\}$. We already know that $\frac{W_p(N_{i,\ell}) - W_d(N_{i,\ell})}{W_p(N_{i,\ell})}$ is non-negative when $\rho > 1$ and $k > z_0$. Because $\nu = R\mu/c > \eta_1 \doteq \max\{z_0, z_4, z_5\} + 1$ implies $k > \max\{z_0, z_4, z_5\}$, part (a) follows.

We now show (EC.131). To do so, we first derive the preliminary inequality (EC.135), which will be used in later steps of the proof. Recall that $N_1 = i + \ell_1 m$ and $N_2 = i + \ell_2 m$, where $\ell_1 < \ell_2$. Then, the balking threshold in the system with N_1 servers is $K_1 = \lfloor \frac{(i+m\ell_1)R\mu}{c} \rfloor = \lfloor \frac{iR\mu}{c} \rfloor + \ell_1 \frac{Rm\mu}{c}$, whereas the corresponding figure in the one with N_2 servers is $K_2 = \lfloor \frac{(i+m\ell_2)R\mu}{c} \rfloor = \lfloor \frac{iR\mu}{c} \rfloor + \ell_2 \frac{Rm\mu}{c}$. Then,

$$\begin{aligned}\frac{K_2}{N_2} - \frac{K_1}{N_1} = \frac{\lfloor \frac{iR\mu}{c} \rfloor + \ell_2 \frac{Rm\mu}{c}}{i + m\ell_2} - \frac{\lfloor \frac{iR\mu}{c} \rfloor + \ell_1 \frac{Rm\mu}{c}}{i + m\ell_1} &= \frac{(m\ell_1 - m\ell_2)\lfloor \frac{iR\mu}{c} \rfloor + (\ell_2 - \ell_1)\frac{iRm\mu}{c}}{(i+m\ell_1)(i+m\ell_2)} \\ &= \frac{(\ell_2 - \ell_1)(\frac{iRm\mu}{c} - m\lfloor \frac{iR\mu}{c} \rfloor)}{(i+m\ell_1)(i+m\ell_2)} \\ &\geq 0. \end{aligned} \tag{EC.134}$$

We can represent K_1 as $K_1 = N_1(k+d)$, where $0 \leq d < 1$. Thus, (EC.134) implies

$$K_2 \geq \frac{N_2}{N_1} K_1 = N_2(k+d). \tag{EC.135}$$

Recall from (EC.11) the average sojourn time in the SQ system. Then, because $K_1 = N_1(k+d)$, in the SQ system with N_1 servers, we have

$$\begin{aligned}
W_s(N_1) &= \frac{1}{N_1\lambda(\rho-1)} \frac{K_1\rho^{K_1+2} - (K_1+1)\rho^{K_1+1} + \rho}{\rho^{K_1} - 1} \\
&= \frac{1}{N_1\lambda(\rho-1)} \frac{N_1(k+d)\rho^{N_1(k+d)+2} - (N_1(k+d)+1)\rho^{N_1(k+d)+1} + \rho}{\rho^{N_1(k+d)} - 1} \\
&= \frac{1}{\lambda(\rho-1)} \frac{(k+d)\rho^{N_1(k+d)+2} - ((k+d)+\frac{1}{N_1})\rho^{N_1(k+d)+1} + \frac{1}{N_1}\rho}{\rho^{N_1(k+d)} - 1} \\
&= \frac{1}{\lambda(\rho-1)} \left((k+d)\rho^2 - \left(k+d+\frac{1}{N_1}\right)\rho + \frac{(k+d)\rho^2 - (k+d)\rho}{\rho^{N_1(k+d)} - 1} \right). \qquad \text{(EC.136)}
\end{aligned}$$

In the SQ system with N_2 servers,

$$\begin{aligned}
W_s(N_2) &= \frac{1}{N_2\lambda(\rho-1)} \frac{K_2\rho^{K_2+2} - (K_2+1)\rho^{K_2+1} + \rho}{\rho^{K_2} - 1} \\
&\geq \frac{1}{N_2\lambda(\rho-1)} \frac{N_2(k+d)\rho^{N_2(k+d)+2} - (N_2(k+d)+1)\rho^{N_2(k+d)+1} + \rho}{\rho^{N_2(k+d)} - 1} \qquad \text{(EC.137)} \\
&= \frac{1}{\lambda(\rho-1)} \frac{(k+d)\rho^{N_2(k+d)+2} - ((k+d)+\frac{1}{N_2})\rho^{N_2(k+d)+1} + \frac{1}{N_2}\rho}{\rho^{N_2(k+d)} - 1} \\
&= \frac{1}{\lambda(\rho-1)} \left((k+d)\rho^2 - \left(k+d+\frac{1}{N_2}\right)\rho + \frac{(k+d)\rho^2 - (k+d)\rho}{\rho^{N_2(k+d)} - 1} \right). \qquad \text{(EC.138)}
\end{aligned}$$

We have (EC.137) because W_s is increasing in K by (EC.53) and $K_2 \geq N_2(k+d)$ from (EC.135). Based on (EC.136) and (EC.138), we have

$$W_s(N_2) - W_s(N_1) \geq \int_{N_1}^{N_2} f_1'(\gamma)d\gamma, \qquad \text{(EC.139)}$$

where $f_1(\gamma) \doteq \frac{1}{\lambda(\rho-1)} \left((k+d)\rho^2 - (k+d+\frac{1}{\gamma})\rho + \frac{(k+d)\rho^2-(k+d)\rho}{\rho^{\gamma(k+d)}-1} \right)$ for $\gamma \geq 1$. Note that when $k > z_4$,

$$\begin{aligned}
f_1'(\gamma) &= \frac{1}{\lambda(\rho-1)} \left(\frac{1}{\gamma^2}\rho - \frac{((k+d)\rho^2 - (k+d)\rho)\rho^{\gamma(k+d)}(k+d)\ln(\rho)}{(\rho^{\gamma(k+d)}-1)^2} \right) \\
&= \frac{1}{\lambda(\rho-1)\gamma^2(\rho^{\gamma(k+d)}-1)^2} \left(\rho(\rho^{\gamma(k+d)}-1)^2 - \gamma^2(k+d)^2(\rho^2-\rho)\rho^{\gamma(k+d)}\ln(\rho) \right) \\
&= \frac{\rho}{\lambda(\rho-1)\gamma^2(\rho^{\gamma(k+d)}-1)^2} \left((\rho^{\gamma(k+d)}-1)^2 - \gamma^2(k+d)^2\rho^{\gamma(k+d)}(\rho-1)\ln(\rho) \right) \\
&> \frac{\rho}{\lambda(\rho-1)\gamma^2(\rho^{\gamma(k+d)}-1)^2} \frac{1}{2}(\rho^{\gamma(k+d)}-1)^2 \qquad \text{(EC.140)} \\
&= \frac{\rho}{2\lambda(\rho-1)\gamma^2}. \qquad \text{(EC.141)}
\end{aligned}$$

The inequality (EC.140) is because when $k > z_4$ and $\gamma \geq 1$, $\gamma(k+d) > z_4$ and $\frac{(\rho^{(k+d)\gamma}-1)^2}{\gamma^2(k+d)^2\rho^{(k+d)\gamma}} > 2(\rho-1)\ln(\rho)$ by (EC.107), thus

$$\begin{aligned}
\gamma^2(k+d)^2\rho^{(k+d)\gamma}(\rho-1)\ln(\rho) - \frac{1}{2}(\rho^{(k+d)\gamma}-1)^2 &= \frac{1}{2}\gamma^2(k+d)^2\rho^{(k+d)\gamma}\left(2(\rho-1)\ln(\rho) - \frac{(\rho^{(k+d)\gamma}-1)^2}{\gamma^2(k+d)^2\rho^{(k+d)\gamma}} \right) \\
&< 0.
\end{aligned}$$

It follows from (EC.139) and (EC.141) that if $\rho > 1$ and $k > z_4$

$$\begin{aligned}
W_s(N_2) - W_s(N_1) &> \int_{N_1}^{N_2} \frac{\rho}{2\lambda(\rho-1)\gamma^2}d\gamma = \frac{\rho}{2\lambda(\rho-1)}\left(\frac{1}{N_1} - \frac{1}{N_2}\right) \geq \frac{\rho}{2\lambda(\rho-1)}\left(\frac{1}{N_1} - \frac{1}{N_1+1}\right) \\
&= \frac{\rho-1}{\lambda N_1(N_1+1)} \frac{\rho}{2(\rho-1)^2},
\end{aligned}$$

which proves the inequality in (EC.131).

We now show (EC.132). Recall from (8) and (EC.11) the average sojourn time in the pooled and SQ systems, respectively. Then, if $\rho > 1$ and $k > z_5$,

$$
\begin{aligned}
&W_p(N_1) - W_s(N_1) \\
=&\frac{\sum_{j=0}^{N_1-1}\frac{N_1^j}{j!}j\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1} j\rho^j}{\left(\sum_{j=0}^{N_1-1}\frac{N_1^j}{j!}\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j\right)N_1\lambda} - \frac{\sum_{j=0}^{K_1} j\rho^j}{\left(\sum_{j=0}^{K_1-1}\rho^j\right)N_1\lambda} \\
=&\frac{\sum_{j=0}^{N_1-1}\frac{N_1^j}{j!}j\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1} j\rho^j}{\left(\sum_{j=0}^{N_1-1}\frac{N_1^j}{j!}\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j\right)N_1\lambda} - \frac{\sum_{j=0}^{N_1-1}\frac{N_1^{N_1}}{N_1!}j\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1} j\rho^j}{\left(\sum_{j=0}^{N_1-1}\frac{N_1^{N_1}}{N_1!}\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j\right)N_1\lambda} \\
<&\frac{\sum_{j=0}^{N_1-1}\frac{N_1^{N_1}}{N_1!}j\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1} j\rho^j}{\left(\sum_{j=0}^{N_1-1}\frac{N_1^j}{j!}\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j\right)N_1\lambda} - \frac{\sum_{j=0}^{N_1-1}\frac{N_1^{N_1}}{N_1!}j\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1} j\rho^j}{\left(\sum_{j=0}^{N_1-1}\frac{N_1^{N_1}}{N_1!}\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j\right)N_1\lambda} && \text{(EC.142)} \\
=&\frac{\sum_{j=0}^{N_1-1}\frac{N_1^{N_1}}{N_1!}j\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1} j\rho^j}{N_1\lambda}\frac{\sum_{j=0}^{N_1-1}(\frac{N_1^{N_1}}{N_1!} - \frac{N_1^j}{j!})\rho^j}{\left(\sum_{j=0}^{N_1-1}\frac{N_1^j}{j!}\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j\right)\left(\sum_{j=0}^{N_1-1}\frac{N_1^{N_1}}{N_1!}\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j\right)} \\
=&\frac{\sum_{j=0}^{K_1} j\rho^j}{\left(\sum_{j=0}^{K_1-1}\rho^j\right)N_1\lambda}\frac{\sum_{j=0}^{N_1-1}(\frac{N_1^{N_1}}{N_1!} - \frac{N_1^j}{j!})\rho^j}{\left(\sum_{j=0}^{N_1-1}\frac{N_1^j}{j!}\rho^j + \frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j\right)} \\
<&\frac{K_1}{N_1\mu}\frac{N_1(\frac{N_1^{N_1}}{N_1!})\rho^{N_1-1}}{\frac{N_1^{N_1}}{N_1!}\sum_{j=N_1}^{K_1-1}\rho^j} && \text{(EC.143)} \\
=&\frac{K_1}{N_1\mu}\frac{N_1\rho^{N_1-1}}{(\rho^{K_1} - \rho^{N_1})/(\rho-1)} \\
<&\frac{k+1}{\mu}\frac{N_1(\rho-1)}{\rho(\rho^{K_1-N_1}-1)} && \text{(EC.144)} \\
=&\frac{k+1}{\lambda}\frac{N_1(\rho-1)}{\rho^{K_1-N_1}-1} \\
\le&\frac{k+1}{\lambda}\frac{N_1(\rho-1)}{\rho^{N_1k-N_1}-1} && \text{(EC.145)} \\
=&\frac{\rho-1}{\lambda N_1(N_1+1)}\left(\frac{N_1^2(N_1+1)(k+1)}{\rho^{N_1k-N_1}-1}\right) \\
\le&\frac{\rho-1}{\lambda N_1(N_1+1)}\left(\frac{2(k+1)}{\rho^{k-1}-1}\right) && \text{(EC.146)} \\
<&\frac{\rho-1}{\lambda N_1(N_1+1)}\frac{\rho}{2(\rho-1)^2}, && \text{(EC.147)}
\end{aligned}
$$

which proves the inequality in (EC.132). Let us explain why each of the inequalities in (EC.142) through (EC.147) holds. The inequality (EC.142) is because $\frac{N_1^j}{j!} < \frac{N_1^{N_1}}{N_1!}$ for $j \in \{1, 2, \ldots, N_1 - 2\}$ and $\frac{N_1^j}{j!} = \frac{N_1^{N_1}}{N_1!}$ for $j = N_1 - 1$. The inequality (EC.143) is because

$$
\frac{\sum_{j=0}^{K_1} j\rho^j}{\left(\sum_{j=0}^{K_1-1}\rho^j\right)N_1\lambda} \le \frac{K_1\sum_{j=1}^{K_1}\rho^j}{\left(\sum_{j=0}^{K_1-1}\rho^j\right)N_1\lambda} = \frac{K_1}{N_1\mu}
$$

and for $\rho > 1$ and $j \in \{1,2,\ldots,N_1-1\}$,

$$\left(\frac{N_1^{N_1}}{N_1!} - \frac{N_1^j}{j!}\right)\rho^j < \frac{N_1^{N_1}}{N_1!}\rho^{N_1-1}.$$

The inequality (EC.144) follows from the fact that

$$\frac{K_1}{N_1\mu} < \frac{(k+1)N_1}{N_1\mu} = \frac{k+1}{\mu}.$$

We have (EC.145) because $K_1 \geq N_1 k$ and $\rho > 1$. The inequality (EC.146) is due to the fact that $f_2(\gamma) \doteq \frac{\gamma^2(\gamma+1)(k+1)}{\rho^{\gamma k-\gamma}-1}$ is decreasing in γ when $k > z_5$ and $\gamma \geq 1$, which is shown below, and hence $\frac{N_1^2(N_1+1)(k+1)}{\rho^{N_1 k - N_1}-1} \leq \frac{2(k+1)}{\rho^{k-1}-1}$. When $k > z_5$, we have $k > \frac{3}{\ln(\rho)} + 1$, and thus

$$\begin{aligned}
f_2'(\gamma) =&(k+1)\frac{(3\gamma^2+2\gamma)(\rho^{\gamma k-\gamma}-1) - \gamma^2(\gamma+1)(\rho^{\gamma k-\gamma})(k-1)\ln(\rho)}{(\rho^{\gamma k-\gamma}-1)^2}\\
<&(k+1)\frac{3\gamma^2(\gamma+1)(\rho^{\gamma k-\gamma}) - \gamma^2(\gamma+1)(\rho^{\gamma k-\gamma})(k-1)\ln(\rho)}{(\rho^{\gamma k-\gamma}-1)^2}\\
=&(k+1)\frac{\gamma^2(\gamma+1)(\rho^{\gamma k-\gamma})(3-(k-1)\ln(\rho))}{(\rho^{\gamma k-\gamma}-1)^2}\\
<&0.
\end{aligned}$$

Finally, the inequality (EC.147) is because when $\rho > 1$ and $k > z_5$, $\frac{\rho}{2(\rho-1)^2} > \frac{2(k+1)}{\rho^{k-1}-1}$ by (EC.108). $\square$

Proof of Theorem 2 - Part (b): Take any $i \in \{1,2,\ldots,m\}$, and consider the system size subsequence $\{N_{i,\ell} = i + \ell m, \ell = 0,1,\ldots\}$. Recall the definition of η_2 from (EC.105). When $N_{i,\ell} = 1$, $\frac{SW_d(i+m\ell)-SW_p(i+m\ell)}{SW_p(i+m\ell)} = 0$. For any $N_{i,\ell} \geq 2$, according to the proof of part (a), the conditions in Theorem 1-(a) are satisfied and thus $\frac{SW_d(i+m\ell)-SW_p(i+m\ell)}{SW_p(i+m\ell)} > 0$ when $k > z_0$ and $\rho > 1$. As a result, the percentage subsequence for social welfare is non-negative under the conditions stated in part (b).

We now show that if $\rho > 1$ and $k > \max\{z_6, z_7, z_8\}$, then $\frac{SW_d(i+m\ell)-SW_p(i+m\ell)}{SW_p(i+m\ell)}$ is strictly increasing in ℓ, $\ell \in \mathbb{N}$. By the definition of the subsequence $\{N_{i,\ell}, \ell = 0,1,\ldots\}$, the system size is equal to $i + \ell m$. Then the balking threshold in the pooled system is

$$K = d_i + \ell d_m, \tag{EC.148}$$

where

$$d_i \doteq \lfloor Ri\mu/c \rfloor \quad \text{and} \quad d_m = Rm\mu/c. \tag{EC.149}$$

Throughout this proof, we will include the system size as an argument of $SW_d(\cdot)$, $SW_s(\cdot)$ and $SW_p(\cdot)$; here, the index $j = s$ represents the SQ system. The following lemma identifies a bound, which we will use later in the proof.

LEMMA EC.9. *If $\rho > 1$ and $k > \max\{z_6, z_7\}$,*

$$\frac{SW_d(i+m(\ell+1)) - SW_s(i+m(\ell+1))}{SW_s(i+m(\ell+1))} - \frac{SW_d(i+\ell m) - SW_s(i+\ell m)}{SW_s(i+\ell m)} > \frac{SW_d(i+\ell m)}{SW_s(i+\ell m)}\frac{1}{2(\ell+2)}.$$

Proof of Lemma EC.9: Recall from Lemmas 2 and EC.1 that

$$SW_d(i+\ell m) = \left(\frac{1-\rho^k}{1-\rho^{k+1}}R\lambda - \frac{\rho-(k+1)\rho^{k+1}+k\rho^{k+2}}{(\rho-1)(\rho^{k+1}-1)}c\right)(i+\ell m), \tag{EC.150}$$

and

$$SW_s(i+\ell m) = \frac{1-\rho^{d_i+\ell d_m}}{1-\rho^{d_i+\ell d_m+1}} R(i+\ell m)\lambda - \frac{\rho-(d_i+\ell d_m+1)\rho^{d_i+\ell d_m+1}+(d_i+\ell d_m)\rho^{d_i+\ell d_m+2}}{(\rho-1)(\rho^{d_i+\ell d_m+1}-1)}c. \quad \text{(EC.151)}$$

We will prove this lemma under each of the two possible cases about ℓ.

Case 1: First, we prove Lemma EC.9 for $\ell = 0$, i.e.,

$$\frac{SW_d(i+m)-SW_s(i+m)}{SW_s(i+m)} - \frac{SW_d(i)-SW_s(i)}{SW_s(i)} > \frac{SW_d(i)}{SW_s(i)}\frac{1}{4},$$

which is equivalent to

$$\frac{SW_d(i+m)}{SW_d(i)} > \frac{5}{4}\frac{SW_s(i+m)}{SW_s(i)}. \quad \text{(EC.152)}$$

Using (EC.150) and the fact that $1 \le i \le m$,

$$\frac{SW_d(i+m)}{SW_d(i)} = \frac{i+m}{i} \ge 2. \quad \text{(EC.153)}$$

Recall the definition of d_i and d_m from (EC.149), define

$$\bar{r}_i \doteq \frac{Ri\mu}{c} - d_i. \quad \text{(EC.154)}$$

Then $0 \le \bar{r}_i < 1$. Since $\frac{Rm\mu}{c}$ is an integer, then $\frac{R(i+\ell m)\mu}{c} - (d_i+\ell d_m) = \bar{r}_i$ for $\ell \in \mathbb{N}$.

Define $r_2 \doteq \frac{RN\mu}{c} - K$. Then, the social welfare for the scaled queuing system, that is, the M/M/1/K system described in Lemma EC.1, can be expressed as follows.

$$SW_s(N) = \frac{c}{(\rho^{K+1}-1)(\rho-1)}\left(r_2\rho^{K+2}+(1-r_2)\rho^{K+1}-(K+r_2)\rho^2+(K+r_2-1)\rho\right). \quad \text{(EC.155)}$$

Replacing K with $d_i+\ell d_m$ and r_2 with $\bar{r}_i$ in the above expression for $SW_s(N=i+\ell m)$, we have

$$SW_s(i+\ell m) = \frac{c}{(\rho^{d_i+\ell d_m+1}-1)(\rho-1)}\left(\bar{r}_i\rho^{d_i+\ell d_m+2}+(1-\bar{r}_i)\rho^{d_i+\ell d_m+1}-(d_i+\ell d_m+\bar{r}_i)\rho^2+(d_i+\ell d_m+\bar{r}_i-1)\rho\right).$$

Then, for $\ell = 0$, we have the following when $\rho > 1$ and $k > z_6$:

$$\begin{aligned}
SW_s(i) =& \frac{c}{(\rho^{d_i+1}-1)(\rho-1)}\left(\bar{r}_i\rho^{d_i+2}+(1-\bar{r}_i)\rho^{d_i+1}-(d_i+\bar{r}_i)\rho^2+(d_i+\bar{r}_i-1)\rho\right)\\
=& \frac{c}{\rho-1}\left(\bar{r}_i\rho+(1-\bar{r}_i)+\frac{\bar{r}_i\rho+(1-\bar{r}_i)-(d_i+\bar{r}_i)\rho^2+(d_i+\bar{r}_i-1)\rho}{\rho^{d_i+1}-1}\right)\\
>& \frac{c}{\rho-1}\left(\bar{r}_i\rho+(1-\bar{r}_i)+\frac{\bar{r}_i\rho+(1-\bar{r}_i)-(d_i+1)\rho^2}{\rho^{d_i+1}-1}\right)\\
>& \frac{c}{\rho-1}\left(\frac{5}{8}(\bar{r}_i\rho+(1-\bar{r}_i))+\frac{\bar{r}_i\rho+(1-\bar{r}_i)}{\rho^{d_i+1}-1}\right) \qquad \text{(EC.156)}\\
>& \frac{5c}{8(\rho-1)}\left(\bar{r}_i\rho+(1-\bar{r}_i)+\frac{\bar{r}_i\rho+(1-\bar{r}_i)}{\rho^{d_i+1}-1}\right). \qquad \text{(EC.157)}
\end{aligned}$$

The inequality (EC.156) is because $\frac{(d_i+1)\rho^2}{\rho^{d_i+1}-1} < \frac{3}{8} < \frac{3}{8}(\bar{r}_i\rho+(1-\bar{r}_i))$ when $\rho > 1$ and $d_i+1 > k > z_6$ since $\frac{z}{\rho^z-1}$ is strictly decreasing when $\rho > 1$ and $z > \frac{1}{\ln(\rho)}$ according to the proof of Lemma EC.8-(c) and $\frac{z_6}{\rho^{z_6}-1} < \frac{3}{8\rho^2}$ according to (EC.109). Moreover,

$$\begin{aligned}
SW_s(i+m) =& \frac{c}{(\rho^{d_i+d_m+1}-1)(\rho-1)}\left(\bar{r}_i\rho^{d_i+d_m+2}+(1-\bar{r}_i)\rho^{d_i+d_m+1}-(d_i+d_m+\bar{r}_i)\rho^2+(d_i+d_m+\bar{r}_i-1)\rho\right)\\
=& \frac{c}{\rho-1}\left(\bar{r}_i\rho+(1-\bar{r}_i)+\frac{\bar{r}_i\rho+(1-\bar{r}_i)-(d_i+d_m+\bar{r}_i)\rho^2+(d_i+d_m+\bar{r}_i-1)\rho}{\rho^{d_i+d_m+1}-1}\right)\\
<& \frac{c}{\rho-1}\left(\bar{r}_i\rho+(1-\bar{r}_i)+\frac{\bar{r}_i\rho+(1-\bar{r}_i)}{\rho^{d_i+d_m+1}-1}\right)\\
<& \frac{c}{\rho-1}\left(\bar{r}_i\rho+(1-\bar{r}_i)+\frac{\bar{r}_i\rho+(1-\bar{r}_i)}{\rho^{d_i+1}-1}\right). \qquad \text{(EC.158)}
\end{aligned}$$

Combing (EC.157) and (EC.158), we have

$$\frac{SW_s(i+m)}{SW_s(i)} < \frac{8}{5}. \tag{EC.159}$$

Combining (EC.153) and (EC.159), (EC.152) follows. Thus, the lemma holds for $\ell = 0$.

Case 2: Now, we focus on the case that $\ell \geq 1$. When $\rho > 1$ and $k > \max\{z_6, z_7\}$,

$$\begin{aligned}
SW_s(i+\ell m) =& \frac{1-\rho^{d_i+\ell d_m}}{1-\rho^{d_i+\ell d_m+1}} R(i+\ell m)\lambda - \frac{\rho - (d_i+\ell d_m+1)\rho^{d_i+\ell d_m+1} + (d_i+\ell d_m)\rho^{d_i+\ell d_m+2}}{(\rho-1)(\rho^{d_i+\ell d_m+1}-1)} c \\
=& \frac{\rho^{d_i+\ell d_m}-1}{\rho^{d_i+\ell d_m+1}-1} R(i+\ell m)\lambda - \left((d_i+\ell d_m)\rho - (d_i+\ell d_m+1) + \frac{(d_i+\ell d_m+1)(\rho-1)}{\rho^{d_i+\ell d_m+1}-1} \right) \frac{c}{\rho-1} \\
=& \left(\left(\frac{1}{\rho} - \frac{1-\frac{1}{\rho}}{\rho^{d_i+\ell d_m+1}-1} \right) \frac{R(i+\ell m)\lambda}{c} - (d_i+\ell d_m) - \frac{d_i+\ell d_m+1}{\rho^{d_i+\ell d_m+1}-1} + \frac{1}{\rho-1} \right) c \\
=& \left(\left(1 - \frac{\rho-1}{\rho^{d_i+\ell d_m+1}-1} \right) \frac{R(i+\ell m)\mu}{c} - (d_i+\ell d_m) - \frac{d_i+\ell d_m+1}{\rho^{d_i+\ell d_m+1}-1} + \frac{1}{\rho-1} \right) c.
\end{aligned} \tag{EC.160}$$

To get a preliminary bound, we now assume that $\ell \in \mathbb{R}_+$ and $\ell \geq 1$, which imply that the social welfare is a continuous function of ℓ. Later, we will eliminate that assumption to focus on $\ell \in \mathbb{N}_+$ and use this preliminary bound to prove the statement in Lemma EC.9. Taking the derivative of (EC.160), we get:

$$\begin{aligned}
\frac{\partial SW_s(i+\ell m)}{\partial \ell} =& \left(\left(1 - \frac{\rho-1}{\rho^{d_i+\ell d_m+1}-1} \right) \frac{Rm\mu}{c} + \frac{\rho-1}{(\rho^{d_i+\ell d_m+1}-1)^2} \rho^{d_i+\ell d_m+1} d_m \ln(\rho) \frac{R(i+\ell m)\mu}{c} - d_m \right) c \\
& - \left(\frac{d_m(\rho^{d_i+\ell d_m+1}-1) - (d_i+\ell d_m+1)\rho^{d_i+\ell d_m+1} d_m \ln(\rho)}{(\rho^{d_i+\ell d_m+1}-1)^2} \right) c \\
<& \left(\left(1 - \frac{\rho-1}{\rho^{d_i+\ell d_m+1}-1} \right) d_m + \frac{\rho-1}{(\rho^{d_i+\ell d_m+1}-1)^2} \rho^{d_i+\ell d_m+1} d_m \ln(\rho)(d_i+\ell d_m+1) - d_m \right) c \\
& - \left(\frac{d_m(\rho^{d_i+\ell d_m+1}-1) - (d_i+\ell d_m+1)\rho^{d_i+\ell d_m+1} d_m \ln(\rho)}{(\rho^{d_i+\ell d_m+1}-1)^2} \right) c & \text{(EC.161)} \\
<& \left(\frac{\rho}{(\rho^{d_i+\ell d_m+1}-1)^2} \rho^{d_i+\ell d_m+1} d_m \ln(\rho)(d_i+\ell d_m+1) \right) c \\
=& \left(\frac{\rho^{d_i+\ell d_m+2}(d_i+\ell d_m+1) d_m \ln(\rho)}{(\rho^{d_i+\ell d_m+1}-1)^2} \right) c \\
<& \left(\frac{\rho^{d_i+\ell d_m+2}(d_i+\ell d_m+1)^2 \ln(\rho)}{(\rho^{d_i+\ell d_m+1}-1)^2} \right) c & \text{(EC.162)} \\
\leq& \left(\frac{(d_i+\ell d_m+1)^2 \rho^2 \ln(\rho)}{(\rho-1)(\rho^{d_i+\ell d_m+1}-1)} \right) c & \text{(EC.163)} \\
<& \frac{c}{4(\rho-1)(d_i+\ell d_m+1)} & \text{(EC.164)} \\
\leq& \frac{c}{4(\rho-1)(\ell+1)}. & \text{(EC.165)}
\end{aligned}$$

Here, the inequality (EC.161) follows from the facts that $\frac{Rm\mu}{c} = d_m$ and $\frac{R(i+\ell m)\mu}{c} < K+1 = d_i+\ell d_m+1$. The inequality (EC.162) is because $d_m < d_i+\ell d_m+1$ when $\ell \geq 1$. We have (EC.163) because $\frac{\rho^{d_i+\ell d_m+2}}{\rho^{d_i+\ell d_m+1}-1} \leq \frac{\rho^2}{\rho-1}$. The reason for (EC.164) is as follows. By the proof of Lemma EC.8-(d), if $\rho > 1$ and $k > z_7$, which implies $d_i+\ell d_m+1 > K > z_7$, $\frac{\rho^{d_i+\ell d_m+1}-1}{(d_i+\ell d_m+1)^3} > 4\rho^2\ln(\rho)$, i.e., $\left(\frac{(d_i+\ell d_m+1)^2\rho^2\ln(\rho)}{(\rho^{d_i+\ell d_m+1}-1)} \right) < \frac{1}{4(d_i+\ell d_m+1)}$.

Based on these, we have

$$\frac{\partial}{\partial \ell} \left(\frac{SW_d(i+\ell m) - SW_s(i+\ell m)}{SW_s(i+\ell m)} \right) = \frac{\partial}{\partial \ell} \left(\frac{SW_d(i+\ell m)}{SW_s(i+\ell m)} \right)$$

$$
\begin{aligned}
&= \frac{\frac{\partial SW_d(i+\ell m)}{\partial \ell} SW_s(i+\ell m) - SW_d(i+\ell m)\frac{\partial SW_s(i+\ell m)}{\partial \ell}}{(SW_s(i+\ell m))^2} \\
&= \frac{SW_d(i+\ell m)\frac{m}{i+\ell m} SW_s(i+\ell m) - SW_d(i+\ell m)\frac{\partial SW_s(i+\ell m)}{\partial \ell}}{(SW_s(i+\ell m))^2} \\
&= SW_d(i+\ell m)\frac{\frac{m}{i+\ell m} SW_s(i+\ell m) - \frac{\partial SW_s(i+\ell m)}{\partial \ell}}{(SW_s(i+\ell m))^2} \\
&= \frac{SW_d(i+\ell m)}{SW_s(i+\ell m)}\left(\frac{m}{i+\ell m} - \frac{\partial SW_s(i+\ell m)}{\partial \ell}/SW_s(i+\ell m)\right) \\
&\geq \frac{SW_d(i+\ell m)}{SW_s(i+\ell m)}\left(\frac{1}{\ell+1} - \frac{\partial SW_s(i+\ell m)}{\partial \ell}/SW_s(i+\ell m)\right) \qquad \text{(EC.166)} \\
&> \frac{SW_d(i+\ell m)}{SW_s(i+\ell m)}\left(\frac{1}{\ell+1} - \frac{c}{4(\rho-1)(\ell+1)}\Big/\frac{c}{2(\rho-1)}\right) \qquad \text{(EC.167)} \\
&= \frac{SW_d(i+\ell m)}{SW_s(i+\ell m)}\left(\frac{1}{2(\ell+1)}\right). \qquad \text{(EC.168)}
\end{aligned}
$$

The above inequality in (EC.166) is because $i \leq m$, and the inequality in (EC.167) follows from (EC.165) and (EC.170), which will be shown below. We now show (EC.170). We have

$$
\begin{aligned}
SW_s(i+\ell m) &\geq \left(\left(1 - \frac{\rho-1}{\rho^{d_i+\ell d_m+1}-1}\right)(d_i+\ell d_m) - (d_i+\ell d_m) - \frac{d_i+\ell d_m+1}{\rho^{d_i+\ell d_m+1}-1} + \frac{1}{\rho-1}\right)c \qquad \text{(EC.169)} \\
&= \left(\frac{1}{\rho-1} - \frac{(\rho-1)(d_i+\ell d_m)}{\rho^{d_i+\ell d_m+1}-1} - \frac{(d_i+\ell d_m+1)}{\rho^{d_i+\ell d_m+1}-1}\right)c \\
&> \left(\frac{1}{\rho-1} - \frac{(d_i+\ell d_m+1)\rho}{\rho^{d_i+\ell d_m+1}-1}\right)c \\
&> \frac{c}{2(\rho-1)}. \qquad \text{(EC.170)}
\end{aligned}
$$

Here, the inequality (EC.169) follows because (EC.160) holds and $\frac{R(i+\ell m)\mu}{c} \geq d_i + \ell d_m$, and the reason for (EC.170) is as follows. By the proof of Lemma EC.8-(c), $\frac{z}{\rho^z-1}$ is strictly decreasing when $\rho > 1$ and $z > \frac{1}{\ln(\rho)}$. Also, $\frac{z_6}{\rho^{z_6}-1} \leq \frac{3}{8\rho^2} < \frac{1}{2\rho(\rho-1)}$ according to (EC.109). Thus, when $\rho > 1$ and $k > z_6$, $\frac{(d_i+\ell d_m+1)\rho}{\rho^{d_i+\ell d_m+1}-1} < \frac{1}{2(\rho-1)}$ since $k > z_6$ implies $d_i + \ell d_m + 1 > z_6$. (It is obvious that $d_i \geq k$ since $i \geq 1$.)

Based on the analysis above, we now show the statement in the lemma. If $\rho > 1$ and $k > \max\{z_6, z_7\}$,

$$
\begin{aligned}
&\frac{SW_d(i+m(\ell+1)) - SW_s(i+m(\ell+1))}{SW_s(i+m(\ell+1))} - \frac{SW_d(i+\ell m) - SW_s(i+\ell m)}{SW_s(i+\ell m)} \\
&> \int_\ell^{\ell+1} \frac{SW_d(i+mx)}{SW_s(i+mx)}\frac{1}{2(x+1)}dx \qquad \text{(EC.171)} \\
&> \int_\ell^{\ell+1} \frac{SW_d(i+\ell m)}{SW_s(i+\ell m)}\frac{1}{2(\ell+2)}dx \qquad \text{(EC.172)} \\
&= \frac{SW_d(i+\ell m)}{SW_s(i+\ell m)}\frac{1}{2(\ell+2)}. \qquad \text{(EC.173)}
\end{aligned}
$$

Above, (EC.171) follows from (EC.168); (EC.172) is because $\frac{SW_d(i+mx)}{SW_s(i+mx)} \geq \frac{SW_d(i+m\ell)}{SW_s(i+m\ell)}$ as $\frac{SW_d(i+mx)}{SW_s(i+mx)}$ is increasing in x for $\rho > 1$, $k > z_7$, and $\ell \leq x \leq \ell + 1$ by (EC.168). These complete the proof of Lemma EC.9. □

For any system size N, we have

$$
SW_s(N) - SW_p(N) = \left(\frac{\frac{N^N}{N!}\rho^K}{\sum_{j=0}^{N-1}\frac{N^j}{j!}\rho^j + \sum_{j=N}^{K}\frac{N^N}{N!}\rho^j} - \frac{\frac{N^N}{N!}\rho^K}{\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j + \sum_{j=N}^{K}\frac{N^N}{N!}\rho^j}\right)RN\lambda
$$

$$
\begin{aligned}
&+\left(\frac{\sum_{j=0}^{N-1}\frac{N^j}{j!}j\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}j\rho^j}{\sum_{j=0}^{N-1}\frac{N^j}{j!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j}-\frac{\sum_{j=0}^{N-1}\frac{N^N}{N!}j\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}j\rho^j}{\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j}\right)c\\
&<\frac{N^N}{N!}\rho^K\left(\frac{\sum_{j=0}^{N-1}(\frac{N^N}{N!}-\frac{N^j}{j!})\rho^j}{(\sum_{j=0}^{N-1}\frac{N^j}{j!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)(\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)}\right)RN\lambda\\
&+\left(\sum_{j=0}^{N-1}\frac{N^N}{N!}j\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}j\rho^j\right)\left(\frac{\sum_{j=0}^{N-1}(\frac{N^N}{N!}-\frac{N^j}{j!})\rho^j}{(\sum_{j=0}^{N-1}\frac{N^j}{j!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)(\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)}\right)c\\
&<\left(\frac{\sum_{j=0}^{N-1}(\frac{N^N}{N!}-\frac{N^j}{j!})\rho^j}{(\sum_{j=0}^{N-1}\frac{N^j}{j!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)(\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)}\frac{N^N}{N!}\rho^K(K+1)\rho\right)c\\
&+\left(\left(\sum_{j=0}^{N-1}\frac{N^N}{N!}j\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}j\rho^j\right)\frac{\sum_{j=0}^{N-1}(\frac{N^N}{N!}-\frac{N^j}{j!})\rho^j}{(\sum_{j=0}^{N-1}\frac{N^j}{j!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)(\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)}\right)c\\
&<\left(\frac{\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j}{(\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)^2}\frac{N^N}{N!}(K+1)\rho^{K+1}\right)c\\
&+\left(\sum_{j=0}^{N-1}\frac{N^N}{N!}j\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}j\rho^j\right)\frac{\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j}{(\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)(\sum_{j=0}^{N-1}\frac{N^N}{N!}\rho^j+\sum_{j=N}^{K}\frac{N^N}{N!}\rho^j)}c\\
&<\left(\frac{\sum_{j=0}^{N-1}\rho^j}{(\sum_{j=N}^{K}\rho^j)^2}(K+1)\rho^{K+1}+K\frac{\sum_{j=0}^{N-1}\rho^j}{\sum_{j=N}^{K}\rho^j}\right)c\\
&=\left((K+1)\rho^{K+1}\frac{(\rho^N-1)(\rho-1)}{(\rho^{K+1}-\rho^N)^2}+K\frac{\rho^N-1}{\rho^{K+1}-\rho^N}\right)c. \qquad \text{(EC.174)}
\end{aligned}
$$

Based on this, we have the following lemma.

LEMMA EC.10. *Recall that $N=i+\ell m$ and $K=d_i+\ell d_m$. Then, for $\rho>1$ and $k>z_8$,*

$$SW_s(i+\ell m)-SW_p(i+\ell m)<\frac{c}{2(\rho-1)}\frac{1}{2\ell+5}.$$

Proof of Lemma EC.10: We have the the following for $\rho>1$ and $k>z_8$:

$$
\begin{aligned}
&SW_s(i+\ell m)-SW_p(i+\ell m)\\
&<\left((d_i+\ell d_m+1)\rho^{d_i+\ell d_m+1}\frac{(\rho^{i+\ell m}-1)(\rho-1)}{(\rho^{d_i+\ell d_m+1}-\rho^{i+\ell m})^2}+(d_i+\ell d_m)\frac{\rho^{i+\ell m}-1}{\rho^{d_i+\ell d_m+1}-\rho^{i+\ell m}}\right)c \qquad \text{(EC.175)}\\
&<\left((d_i+\ell d_m+1)\rho^{d_i+\ell d_m+1}\frac{(\rho^{i+\ell m}-1)\rho}{(\rho^{d_i+\ell d_m+1}-\rho^{i+\ell m})^2}\right)c\\
&<\left((d_i+\ell d_m+1)\rho^{d_i+\ell d_m+2}\frac{\rho^{i+\ell m}}{(\rho^{d_i+\ell d_m+1}-\rho^{i+\ell m})^2}\right)c\\
&\le\left(((k+1)(i+\ell m)+1)\rho^{(k+1)(i+\ell m)+2}\frac{\rho^{i+\ell m}}{(\rho^{k(i+\ell m)+1}-\rho^{i+\ell m})^2}\right)c \qquad \text{(EC.176)}\\
&=((k+1)(i+m\ell)+1)\frac{\rho^{(k+1)(i+m\ell)+2}}{(\rho^{k(i+m\ell)+1}-\rho^{i+m\ell})}\frac{\rho^{i+m\ell}}{(\rho^{k(i+m\ell)+1}-\rho^{i+m\ell})}c\\
&=((k+1)(i+m\ell)+1)\rho^{i+m\ell+1}\frac{1}{1-\rho^{(i+m\ell)(1-k)-1}}\frac{1}{\rho^{(k-1)(i+m\ell)+1}-1}c\\
&<((k+1)(i+m\ell)+1)2\rho^{i+m\ell+1}\frac{1}{\rho^{(k-1)(i+m\ell)+1}-1}c \qquad \text{(EC.177)}\\
&<2((k+1)(i+m\ell)+1)\frac{1}{\rho^{(k-2)(i+m\ell)}-1}c\\
&\le 2(k+2)(i+m\ell)\frac{1}{\rho^{(k-2)(i+m\ell)}-1}c \qquad \text{(EC.178)}
\end{aligned}
$$

$$
\begin{aligned}
=&2(k+2)(i+m\ell)^2\frac{1}{\rho^{(k-2)(i+m\ell)}-1}\frac{c}{i+\ell m}\\
\le&2(k+2)\frac{1}{\rho^{(k-2)}-1}\frac{c}{i+\ell m} &\text{(EC.179)}\\
<&\frac{1}{10(\rho-1)}\frac{c}{i+\ell m} &\text{(EC.180)}\\
\le&\frac{c}{2(\rho-1)}\frac{1}{2\ell+5}. &\text{(EC.181)}
\end{aligned}
$$

We now explain how we obtain the numbered inequalities above. The inequality (EC.175) is due to (EC.174). Because $ki\le d_i<(k+1)i$ and $k\ell m\le \ell d_m<(k+1)\ell m$, (EC.176) follows from the fact that $d_i+\ell d_m\le (k+1)i+(k+1)\ell m$. The inequality (EC.177) is because $\rho^{(i+m\ell)(1-k)-1}\le \rho^{(-k)}<\frac{1}{2}$ when $i+m\ell\ge 1$, $\rho>1$ and $k>z_8\ge \frac{2}{\ln(\rho)}+2$. The inequality (EC.178) is because $i+m\ell\ge 1$. The inequality (EC.179) is due to the fact that $f_3(\gamma)\doteq \gamma^2\frac{1}{\rho^{(k-2)\gamma}-1}$ is strictly decreasing in γ when $\rho>1$, $\gamma\ge 1$ and $k>z_8\ge\frac{2}{\ln(\rho)}+2$ as shown at the end of the proof. By the proof of Lemma EC.8-(e), if $\rho>1$ and $k>z_8$, $2(k+2)\frac{1}{\rho^{(k-2)}-1}<\frac{1}{10(\rho-1)}$ and thus (EC.180) follows. Finally, the inequality (EC.181) is because $5(i+\ell m)\ge 2\ell+5$ when $i\in\{1,\dots,m\}$ and $\ell=0,1,\dots$.

It only remains to prove that $f_3'(\gamma)<0$ assuming that $\gamma\in\mathbb{R}_+$ and $\gamma\ge 1$. Note that

$$
\begin{aligned}
f_3'(\gamma)=&\frac{2\gamma(\rho^{(k-2)\gamma}-1)-\gamma^2\rho^{(k-2)\gamma}(k-2)\ln(\rho)}{(\rho^{(k-2)\gamma-1})^2}\\
<&\gamma\rho^{(k-2)\gamma}\frac{2-\gamma(k-2)\ln(\rho)}{(\rho^{(k-2)\gamma}-1)^2}\\
\le&\gamma\rho^{(k-2)\gamma}\frac{2-(k-2)\ln(\rho)}{(\rho^{(k-2)\gamma}-1)^2} &\text{(EC.182)}\\
<&0.
\end{aligned}
$$

Here, (EC.182) because $\gamma\ge 1$, $\rho>1$ and $k>z_8\ge\frac{2}{\ln(\rho)}+2$. □

We already know from (EC.170) that $SW_s(i+\ell m)>\frac{c}{2(\rho-1)}$ when $\rho>1$ and $k>\max\{z_6,z_7\}$. Combining this with Lemma EC.10, we have the following for $\rho>1$ and $k>\max\{z_6,z_7,z_8\}$:

$$
SW_p(i+\ell m)>SW_s(i+\ell m)-\frac{c}{2(\rho-1)}\frac{1}{2\ell+5}>\frac{c}{2(\rho-1)}-\frac{c}{2(\rho-1)}\frac{1}{2\ell+5}=\frac{c}{2(\rho-1)}\frac{2\ell+4}{2\ell+5}. \tag{EC.183}
$$

This and Lemma EC.10 together imply that for $\rho>1$ and $k>\max\{z_6,z_7,z_8\}$,

$$
\frac{SW_s(i+\ell m)}{SW_p(i+\ell m)}=1+\frac{SW_s(i+\ell m)-SW_p(i+\ell m)}{SW_p(i+\ell m)}<1+\frac{1}{2\ell+4}. \tag{EC.184}
$$

Then, if $\rho>1$ and $k>\max\{z_6,z_7,z_8\}$, we have the following for any $i\in\{1,2,\dots,m\}$ and $\ell\in\mathbb{N}$:

$$
\begin{aligned}
&\frac{SW_d(i+m(\ell+1))-SW_p(i+m(\ell+1))}{SW_p(i+m(\ell+1))}-\frac{SW_d(i+m\ell)-SW_p(i+m\ell)}{SW_p(i+m\ell)}\\
>&\frac{SW_d(i+m(\ell+1))-SW_s(i+m(\ell+1))}{SW_s(i+m(\ell+1))}-\frac{SW_d(i+m\ell)-SW_p(i+m\ell)}{SW_p(i+m\ell)} &\text{(EC.185)}\\
=&\frac{SW_d(i+m(\ell+1))-SW_s(i+m(\ell+1))}{SW_s(i+m(\ell+1))}-\frac{SW_d(i+m\ell)-SW_s(i+m\ell)}{SW_s(i+m\ell)}\\
+&\frac{SW_d(i+m\ell)-SW_s(i+m\ell)}{SW_s(i+m\ell)}-\frac{SW_d(i+m\ell)-SW_p(i+m\ell)}{SW_p(i+m\ell)}\\
>&\frac{SW_d(i+m\ell)}{SW_s(i+m\ell)}\left(\frac{1}{2(\ell+2)}+1-\frac{SW_s(i+m\ell)}{SW_p(i+m\ell)}\right) &\text{(EC.186)}\\
>&0. &\text{(EC.187)}
\end{aligned}
$$

Here, the inequality (EC.185) is because $SW_p(i+m(\ell+1) < SW_s(i+m(\ell+1))$ by Proposition 2. The inequality (EC.186) is due to Lemma EC.9. The inequality (EC.187) follows from (EC.184).

Based on (EC.187), for any given $i \in \mathbb{N}_+$, $\frac{SW_d(N_{i,\ell})-SW_p(N_{i,\ell})}{SW_p(N_{i,\ell})}$ is increasing in ℓ if $\rho > 1$ and $k > \max\{z_6, z_7, z_8\}$. We already know that the subsequence is non-negative when $\rho > 1$ and $k > z_0$. This and the fact that $\nu = R\mu/c > \eta_2 = \max\{z_0, z_6, z_7, z_8\} + 1$ implies $k > \max\{z_0, z_6, z_7, z_8\}$ complete the proof of the claim. □

Appendix I: Proof of Theorem 3

Because $SW_s(R) > SW_p(R)$ by Proposition 2,

$$\frac{SW_d(R)-SW_p(R)}{SW_p(R)} > \frac{SW_d(R)-SW_s(R)}{SW_s(R)}. \tag{EC.188}$$

Define

$$r \doteq \frac{R\mu}{c} - k \qquad \text{and} \qquad r_2 \doteq \frac{RN\mu}{c} - K. \tag{EC.189}$$

Then $0 \le r < 1$, $0 \le r_2 < 1$ and $r_2 = rN - \lfloor rN \rfloor$. Recall from (EC.8) that the social welfare in the dedicated system is

$$\begin{aligned} SW_d(R) &= \left(\frac{1-\rho^k}{1-\rho^{k+1}}\right) RN\lambda - \left(\frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}\right) Nc \\ &= N\frac{\lambda R(\rho^k-1)(\rho-1) - c(\rho-(k+1)\rho^{k+1}+k\rho^{k+2})}{(\rho^{k+1}-1)(\rho-1)} \\ &= \frac{Nc}{(\rho^{k+1}-1)(\rho-1)}\left(\frac{R\mu}{c}\rho(\rho^k-1)(\rho-1) - (\rho-(k+1)\rho^{k+1}+k\rho^{k+2})\right) \\ &= \frac{Nc}{(\rho^{k+1}-1)(\rho-1)}\left((k+r)\rho(\rho^k-1)(\rho-1) - (\rho-(k+1)\rho^{k+1}+k\rho^{k+2})\right) \\ &= \frac{Nc}{(\rho^{k+1}-1)(\rho-1)}\left(r\rho^{k+2} + (1-r)\rho^{k+1} - (k+r)\rho^2 + (k+r-1)\rho\right). \end{aligned} \tag{EC.190}$$

Similarly, the social welfare in the M/M/1/K system described in Lemma EC.1 can be expressed as follows.

$$SW_s(R) = \frac{c}{(\rho^{K+1}-1)(\rho-1)}\left(r_2\rho^{K+2} + (1-r_2)\rho^{K+1} - (K+r_2)\rho^2 + (K+r_2-1)\rho\right). \tag{EC.191}$$

Define a benefit subsequence $\{R_n, n \in \mathbb{N}_+\}$ such that $R_n \doteq \frac{nc}{\mu}$ for $n \in \mathbb{N}_+$. This implies that if $R = R_n$, then $k = n$, $K = Nn$, $r = 0$ and $r_2 = 0$. Thus, (EC.190) and (EC.191) reduce to

$$SW_d(R_n) = \frac{Nc}{(\rho^{n+1}-1)(\rho-1)}\left(\rho^{n+1} - n\rho^2 + (n-1)\rho\right) \tag{EC.192}$$

$$SW_s(R_n) = \frac{c}{(\rho^{Nn+1}-1)(\rho-1)}\left(\rho^{Nn+1} - Nn\rho^2 + (Nn-1)\rho\right). \tag{EC.193}$$

These and (EC.188) imply that for $\rho > 1$

$$\begin{aligned} &\lim_{n\to\infty} \frac{SW_d(R_n)-SW_p(R_n)}{SW_p(R_n)} \\ \ge &\lim_{n\to\infty} \frac{SW_d(R_n)-SW_s(R_n)}{SW_s(R_n)} \\ = &\lim_{n\to\infty} \frac{\frac{Nc}{(\rho^{n+1}-1)(\rho-1)}(\rho^{n+1}-n\rho^2+(n-1)\rho) - \frac{c}{(\rho^{Nn+1}-1)(\rho-1)}(\rho^{Nn+1}-Nn\rho^2+(Nn-1)\rho)}{\frac{c}{(\rho^{Nn+1}-1)(\rho-1)}(\rho^{Nn+1}-Nn\rho^2+(Nn-1)\rho)} \\ = &\frac{\frac{Nc}{\rho-1} - \frac{c}{\rho-1}}{\frac{c}{\rho-1}} = (N-1). \end{aligned}$$

Thus,

$$\max_R \frac{SW_d(R)-SW_p(R)}{SW_p(R)} \ge \max_n \frac{SW_d(R_n)-SW_p(R_n)}{SW_p(R_n)} > (N-2).$$

This and the fact that $\frac{SW_d(R)-SW_p(R)}{SW_p(R)} = \frac{SW_d(R)}{SW_p(R)} - 1$ for any R complete the proof. □

Appendix J: Proof of Proposition 4

Proof of Part (a): In both dedicated and pooled systems, the service fee affects the social welfare only through balking thresholds. In the dedicated system, suppose that the fee that maximizes welfare is f_d^* and the resulting balking threshold is k_d^*. Consider another system called "dedicated help system," which is a variation of the dedicated system. In this system, there are N single-server systems. The total arrival for the system follows Poisson distribution with an arrival rate $N\lambda$, and each customer is routed to a server with probability $\frac{1}{N}$. Thus, the arrival for each single-server system is a Poisson process with rate λ. In the dedicated help system, all servers are homogeneous and service time of each server is exponentially distributed with rate μ. As soon as a single-server system has no customers, the server gets into "help" mode and he randomly chooses another single-server system in which there is at least one customer waiting (in addition to the customer the server of that queue is serving) and starts serving that customer. If there is no such system, the server stays idle until either a customer arrives to that server or a customer arrives to another queue whose server is busy. (If there is more than one server idling in the "help" mode, one of the servers, who will help, can be chosen randomly whenever a customer arrives at the queue of a busy server.) When a server starts helping another server, the help service process is interrupted and canceled altogether, and the customer being served goes back to her original queue either if a customer arrives to the queue of the server who is helping or the server who is being helped finishes his service and becomes available to serve the customer currently being served by the helper server. The new arrival to a server will be accepted if and only if the number of customers which belong to that server (including the ones that are originally routed to the queue of the server but are helped by other servers, and excluding the one (if any) that is under help of the server) n satisfies $n < k_d^*$.

Let $X_d^i(t)$ denote the number of customers that belong to the i^{th} server of the dedicated system at time t and denote by $X_h^i(t)$ the number of customers that belong to the i^{th} server of the dedicated help system at time t. (The latter excludes the customer from other servers helped by the i^{th} server, and includes the customers from the i^{th} queue helped by the other servers.) By sample path comparison, $X_d^i(t) \geq X_h^i(t)$, $t \geq 0$. Let $\lambda_{e,d}^i$ and $\lambda_{e,h}^i$ denote the throughput for the server i in the dedicated system and in the dedicated help system, respectively. Then, we have $\lambda_{e,h}^i \geq \lambda_{e,d}^i$ because if an arrival joins the i^{th} queue of the dedicated system, it implies that the number of customers in the i^{th} dedicated sub-system is less than k_d^*, and so, she will also join the help system since the number of customers that belong to the i^{th} server of the dedicated help system is even smaller. From this and the fact that $X_d^i(t) \geq X_h^i(t)$ $t \geq 0$, it follows that the average sojourn time for the arrivals to server i who join the dedicated help system is smaller than that who join the dedicated system, i.e., $W_h^i \leq W_d^i$.

Denote by SW_h the social welfare under the described dedicated help system. By the throughput and average sojourn time inequalities above, the dedicated help system results in larger social welfare than the dedicated system with fee f_d^*, i.e., $SW_h \geq SW_d^*$.

The dedicated help system can be thought as a kind of pooled system, but with a different admission policy. Because the socially-optimal admission control for the pooled system is a deterministic threshold policy and it can be achieved by setting a service fee, the pooled system with the socially-optimal fee (for the pooled system) will result in larger social welfare than the dedicated help system, i.e., $SW_p^* \geq SW_h$. As a result, $SW_p^* \geq SW_h \geq SW_d^*$. □

Proof of Part (b): Denote by f_d^{**} and f_p^{**} the optimal service fees that maximize the revenue in the dedicated system and the pooled system, respectively. For any fee $f \geq 0$, the revenue of the dedicated system is $RV_d(f) = \theta_d(f)f$ and the revenue of the pooled system is $RV_p(f) = \theta_p(f)f$. Replacing R with $R - f_d^{**}$ in the proof of Proposition 1 and applying the same ideas as in the proof of Proposition 1, one can show that $\theta_p(f_d^{**}) > \theta_d(f_d^{**})$. Thus, $RV_p(f_d^{**}) > RV_d(f_d^{**})$. Then, $RV_p(f_p^{**}) \geq RV_p(f_d^{**}) > RV_d(f_d^{**})$ since f_p^{**} is the fee that maximizes the revenue for the pooled system. □

Appendix K: Explanations and Proofs of Statements in Subsection 4.4

K.1. Preliminary Analysis

Consider the explained unobservable queue setting in Subsection 4.4. In that setting, let $\widehat{SW}_j$ and $\widehat{W}_j$ represent the equilibrium social welfare and average sojourn time in the system $j \in \{d, p\}$, respectively. Moreover, let $\widehat{L}_p$ the long-run average number of customers in the unobservable pooled system, and $\widehat{L}_d$ be the long-run average number of customers in one of the N unobservable dedicated sub-systems in equilibrium. Then,

$$\widehat{SW}_j = \begin{cases} \left(R - c\widehat{W}_j\right)\widehat{\lambda}_{e,j} = R\widehat{\lambda}_{e,j} - c\widehat{L}_j & \text{if } j = p \\ \left(R - c\widehat{W}_j\right)\widehat{\lambda}_{e,j}N = \left(R\widehat{\lambda}_{e,j} - c\widehat{L}_j\right)N & \text{if } j = d, \end{cases} \tag{EC.194}$$

where $\widehat{\lambda}_{e,p}$ is the equilibrium effective arrival rate in the pooled system and $\widehat{\lambda}_{e,d}$ is the equilibrium effective arrival rate in one of the N unobservable dedicated sub-systems.[14]

Lemmas EC.11 and EC.12 in this section identify equilibrium average sojourn time and social welfare for unobservable pooled and dedicated systems.

REMARK EC.1. To present the supplementary results in full generality, we will consider an unobservable system with any fixed fee $f \geq 0$. Obviously, the analysis with $f = 0$ is a special case of the analysis presented here.
Let $\widetilde{W}_j(x)$ represent the average sojourn time in the system $j \in \{d, p\}$ given that the *effective arrival rate* to a queue is x. Then, $\widetilde{W}_d(x) \doteq \infty$ if $x \geq \mu$, and $\widetilde{W}_p(x) \doteq \infty$ if $x \geq N\mu$. In line with (5), we consider a benefit that is not extremely small, i.e.,

$$(R - f)\mu/c > 1. \tag{EC.195}$$

Specifically, the condition in (EC.195) implies that the service is valuable enough that the unobservable system is not empty all the time.

Based on these, there exists a unique symmetric equilibrium such that the *equilibrium joining probability* $\widehat{q}_j$ for $j \in \{d, p\}$ is

$$\widehat{q}_j = \begin{cases} q_j^* & \text{if} \quad \widetilde{W}_j(0) < \frac{R-f}{c} < \widetilde{W}_j(\Lambda_j) \\ 1 & \text{if} \quad \widetilde{W}_j(\Lambda_j) \leq \frac{R-f}{c}, \end{cases} \tag{EC.196}$$

where q_j^* is the unique solution of $R - f - c\widetilde{W}_j(\Lambda_j q_j^*) = 0$ under the stated conditions in the first line of (EC.196).

Let us explain the conditions in (EC.196). The condition $\widetilde{W}_j(\Lambda_j) \leq \frac{R-f}{c}$, which is equivalent to $R - f - c\widetilde{W}_j(\Lambda_j) \geq 0$, means that even if all potential customers join (i.e., the effective arrival rate is equal to the potential arrival rate), each customer gains a non-negative long-run average net benefit by joining. Thus, joining with probability 1 is the unique equilibrium strategy for all customers. We now explain the case with $\widetilde{W}_j(0) < \frac{R-f}{c} < \widetilde{W}_j(\Lambda_j)$ in (EC.196). The

[14] It is perhaps worth noting that (EC.194) and (3) are different as (EC.194) is concerned with the equilibrium performance measures, such as $\widehat{W}_j$, $\widehat{\lambda}_{e,j}$ and $\widehat{L}_j$.

condition $\widetilde{W}_j(0) < \frac{R-f}{c} < \widetilde{W}_j(\Lambda_j)$ implies that if all potential customers join, each joining customer gets a negative long-run average net benefit. Because a customer is better off by balking in that case, joining with probability 1 cannot be an equilibrium strategy. The aforementioned condition also implies that if none of the potential customers join, a customer is better off by joining the queue as its long-run average net benefit would be non-negative in that case. Thus, balking with probability 1 cannot be an equilibrium strategy either. The unique equilibrium strategy is such that the joining probability is the solution of $R - f - c\widetilde{W}_j(\Lambda_j q_j^*) = 0$, which makes the long-run average net benefit zero. Note that (EC.196) does not include the case $\widetilde{W}_j(0) \geq \frac{R-f}{c}$. This is because (EC.195) implies that $\widetilde{W}_j(0) < \frac{R-f}{c}$.

Based on (EC.196), in equilibrium, the effective arrival rate to a queue is

$$\widehat{\lambda}_{e,j} = \Lambda_j \widehat{q}_j, \quad j \in \{d, p\}, \tag{EC.197}$$

and the average sojourn time in the system j is

$$\widehat{W}_j = \widetilde{W}_j(\widehat{\lambda}_{e,j}), \quad j \in \{d, p\}. \tag{EC.198}$$

It is worth noting that $\lambda \widehat{q}_j < \mu$, and thus, in equilibrium, each system $j \in \{d, p\}$ is stable regardless of the fact that $\rho < 1$ or $\rho \geq 1$.

We first state and prove Lemmas EC.11 and EC.12 that will be used in proving results in Section 4.4. For these lemmas, recall the notation $\rho \doteq \lambda/\mu$.

LEMMA EC.11. *Recall Remark EC.1. In the unobservable pooled system, the average sojourn time and social welfare in equilibrium are respectively given by*

$$\widehat{W}_p = \begin{cases} \frac{R-f}{c} & \text{if } \rho < 1 \text{ and } \frac{R-f}{c} < \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} i\rho^i}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} \rho^i\right) N\lambda}, \text{ or } \rho \geq 1, \\ \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} i\rho^i}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} \rho^i\right) N\lambda} & \text{if } \rho < 1 \text{ and } \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} i\rho^i}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} \rho^i\right) N\lambda} \leq \frac{R-f}{c}, \end{cases} \tag{EC.199}$$

$$\widehat{SW}_p = \begin{cases} N\lambda q_p^* f & \text{if } \rho < 1 \text{ and } \frac{R-f}{c} < \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} i\rho^i}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} \rho^i\right) N\lambda}, \text{ or } \rho \geq 1, \\ RN\lambda - c\frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} i\rho^i}{\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} \rho^i} & \text{if } \rho < 1 \text{ and } \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} i\rho^i}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} \rho^i\right) N\lambda} \leq \frac{R-f}{c}, \end{cases} \tag{EC.200}$$

where q_p^* is the equilibrium joining probability and satisfies the following equation under the stated conditions in the first line of $(EC.200)$*:*

$$\widetilde{W}_p(N\lambda q_p^*) = \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i(\lambda q_p^*/\mu)^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} i(\lambda q_p^*/\mu)^i}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} (\lambda q_p^*/\mu)^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} (\lambda q_p^*/\mu)^i\right) N\lambda q_p^*} = \frac{R-f}{c}. \tag{EC.201}$$

REMARK EC.2. By the proof of Lemma EC.11, there exists a unique q_p^* that satisfies (EC.201) if $\rho \geq 1$, or $\rho < 1$ and $\frac{R-f}{c} < \left(\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} i\rho^i\right) / \left(\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} \rho^i\right) N\lambda\right)$.

Proof of Lemma EC.11: Consider the unobservable pooled system, which is an M/M/N queueing system. Recall that the service rate of each server is μ and suppose that the effective arrival rate is $x < N\mu$. Then, the stationary probability distribution of the number of customers in this system is as follows (see Section 7.3.3 of Kulkarni (2010)):

$$\hat{\pi}_0(x) = \left(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho_x^i + \frac{N^N}{N!} \sum_{i=N}^{\infty} \rho_x^i\right)^{-1}, \tag{EC.202}$$

$$\hat{\pi}_i(x) = \hat{\pi}_0(x) N^i \rho_x^i / i! \quad \text{for } i = 1, \ldots, N \quad \text{and} \quad \hat{\pi}_i(x) = \hat{\pi}_0(x) N^N \rho_x^i / N! \quad \text{for } i = N+1, N+2 \ldots \tag{EC.203}$$

where $\rho_x \doteq \frac{x}{N\mu}$. Then, in this system, the long-run average number of customers is

$$\widetilde{L}_p(x) = \sum_{i=0}^{\infty} \hat{\pi}_i(x) i = \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho_x^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho_x^i}{\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho_x^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho_x^i}.$$

Because $\widetilde{W}_p(x) = \widetilde{L}_p(x)/x$ by Little's law, we have

$$\widetilde{W}_p(x) = \sum_{i=0}^{\infty} \hat{\pi}_i(x) i/x = \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho_x^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho_x^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho_x^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho_x^i)x}.$$

Based on this, if the effective arrival rate is Λ_p and $\Lambda_p = N\lambda < N\mu$, which is equivalent to $\rho < 1$,

$$\widetilde{W}_p(\Lambda_p) = \widetilde{W}_p(N\lambda) = \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i)N\lambda}.$$

On the other hand, if the effective arrival rate is Λ_p and $\rho \geq 1$, we have $\widetilde{W}_p(\Lambda_p) = \infty$. Combining these two cases with the fact that $\widetilde{W}_p(0) = 1/\mu$, it follows from (EC.196) and (EC.197) that the effective arrival rate *in equilibrium* is

$$\widehat{\lambda}_{e,p} = \begin{cases} N\lambda q_p^* & \text{if } \rho < 1 \text{ and } \frac{R-f}{c} < \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i)N\lambda}, \text{ or } \rho \geq 1, \\ N\lambda & \text{if } \rho < 1 \text{ and } \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i)N\lambda} \leq \frac{R-f}{c}, \end{cases} \tag{EC.204}$$

where the equilibrium joining probability q_p^* is chosen such that $\widetilde{W}_p(N\lambda q_p^*) = \frac{R-f}{c}$. (It is perhaps worth noting that we did not include the condition $1/\mu < (R-f)/c$ in the first line of (EC.204) because (EC.195) already implies that.) The solution q_p^* exists and is unique if $\widetilde{W}_p(N\lambda) > (R-f)/c$, which is equivalent to the conditions in the first line of (EC.204). The reason is as follows. It is shown at the end of this proof that $\widetilde{W}_p(Ny)$ strictly increases with y for $y < \mu$. We already know that $\widetilde{W}_p(\cdot)$ is a continuous function for $y < \mu$. These and the facts that $\lim_{y\to 0}\widetilde{W}_p(y) = 1/\mu < (R-f)/c$ by (EC.195) and $\widetilde{W}_p(N\lambda) > (R-f)/c$ imply the existence and the uniqueness of q_p^*.

Based on these, by (EC.198), in equilibrium, the long-run average sojourn time is

$$\widehat{W}_p = \begin{cases} \frac{R-f}{c} & \text{if } \rho < 1 \text{ and } \frac{R-f}{c} < \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i)N\lambda}, \text{ or } \rho \geq 1, \\ \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i)N\lambda} & \text{if } \rho < 1 \text{ and } \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i)N\lambda} \leq \frac{R-f}{c}. \end{cases}$$

By Little's Law, $\widehat{L}_p = \widehat{\lambda}_{e,p}\widehat{W}_p$. Therefore, in equilibrium, the long-run average number of customers in the system is

$$\widehat{L}_p = \begin{cases} N\lambda q_p^* \frac{R-f}{c} & \text{if } \rho < 1 \text{ and } \frac{R-f}{c} < \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i)N\lambda}, \text{ or } \rho \geq 1, \\ \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i} & \text{if } \rho < 1 \text{ and } \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i\rho^i}{(\sum_{i=0}^{N-1} \frac{N^i}{i!} \rho^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} \rho^i)N\lambda} \leq \frac{R-f}{c}. \end{cases}$$

Recall the social welfare from (EC.194). Then, using the expressions above, (EC.200) immediately follow.

We now show that $\widetilde{W}_p(Ny)$ is strictly increasing with y. Note that

$$\begin{aligned} \widetilde{W}_p(Ny) = \frac{\sum_{i=0}^{N-1} \frac{N^i}{i!} i(y/\mu)^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} i(y/\mu)^i}{\left(\sum_{i=0}^{N-1} \frac{N^i}{i!} (y/\mu)^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} (y/\mu)^i\right)Ny} &= \frac{1}{N\mu}\frac{\sum_{i=1}^{N-1} \frac{N^i}{(i-1)!} (y/\mu)^{i-1} + \frac{N^N}{N!}\sum_{i=N}^{\infty} i(y/\mu)^{i-1}}{\sum_{i=0}^{N-1} \frac{N^i}{i!} (y/\mu)^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} (y/\mu)^i} \\ &= \frac{1}{N\mu}\frac{N\sum_{i=0}^{N-2} \frac{N^i}{i!} (y/\mu)^i + \frac{N^N}{N!}\sum_{i=N-1}^{\infty} (i+1)(y/\mu)^i}{\sum_{i=0}^{N-1} \frac{N^i}{i!} (y/\mu)^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} (y/\mu)^i} \\ &= \frac{1}{N\mu}\left(N + \frac{\frac{N^N}{N!}\sum_{i=N}^{\infty} (i+1-N)(y/\mu)^i}{\sum_{i=0}^{N-1} \frac{N^i}{i!} (y/\mu)^i + \frac{N^N}{N!}\sum_{i=N}^{\infty} (y/\mu)^i}\right) \end{aligned}$$

$$= \frac{1}{N\mu}\left(N + \frac{\frac{N^N}{N!}\frac{(y/\mu)^N}{(1-y/\mu)^2}}{\sum_{i=0}^{N-1}\frac{N^i}{i!}(y/\mu)^i + \frac{N^N}{N!}\frac{(y/\mu)^N}{1-y/\mu}}\right)$$
$$= \frac{1}{N\mu}\left(N + \frac{\frac{N^N}{N!}}{\sum_{i=0}^{N-1}\frac{N^i}{i!}(y/\mu)^{i-N}(1-y/\mu)^2 + \frac{N^N}{N!}(1-y/\mu)}\right).$$

Because $\sum_{i=0}^{N-1}\frac{N^i}{i!}(y/\mu)^{i-N}(1-y/\mu)^2 + \frac{N^N}{N!}(1-y/\mu)$ is strictly decreasing in y for $y<\mu$, $\widetilde{W}_p(Ny)$ is strictly increasing in y. □

LEMMA EC.12. *Recall Remark EC.1. In the unobservable dedicated system, the average sojourn time and social welfare in equilibrium are respectively given as*

$$\widehat{W}_d = \begin{cases} \frac{R-f}{c} & \text{if } \rho<1 \text{ and } \frac{R-f}{c} < \frac{1}{\mu-\lambda}, \text{ or } \rho\geq 1 \\ \frac{1}{\mu-\lambda} & \text{if } \rho<1 \text{ and } \frac{1}{\mu-\lambda} \leq \frac{R-f}{c}, \end{cases} \tag{EC.205}$$

$$\widehat{SW}_d = \begin{cases} \left(\mu - \frac{c}{R-f}\right) fN & \text{if } \rho<1 \text{ and } \frac{R-f}{c} < \frac{1}{\mu-\lambda}, \text{ or } \rho\geq 1 \\ \left(R\lambda - \frac{c\lambda}{\mu-\lambda}\right) N & \text{if } \rho<1 \text{ and } \frac{1}{\mu-\lambda} \leq \frac{R-f}{c}. \end{cases} \tag{EC.206}$$

Proof of Lemma EC.12: Each unobservable dedicated queue is an M/M/1 queue with service rate μ. Suppose that the effective arrival rate in a queue is $x<\mu$. Then, by Section 7.3.1 of Kulkarni (2010), the average sojourn time is $\widetilde{W}_d(x) = \frac{1}{\mu-x}$ and the average number of customers in one of the N separate sub-systems is $\widetilde{L}_d(x) = \frac{x}{\mu-x}$. Based on this,

$$\widetilde{W}_d(\Lambda_d) = \widetilde{W}_d(\lambda) = \begin{cases} \frac{1}{\mu-\lambda} & \text{if } \lambda<\mu \\ \infty & \text{if } \lambda\geq\mu. \end{cases}$$

Using (EC.196) through (EC.198), the equilibrium effective arrival rate in each queue is

$$\widehat{\lambda}_{e,d} = \begin{cases} \lambda q_d^* = \mu - c/(R-f) & \text{if } \rho<1 \text{ and } \frac{R-f}{c} < \frac{1}{\mu-\lambda}, \text{ or } \rho\geq 1 \\ \lambda & \text{if } \rho<1 \text{ and } \frac{1}{\mu-\lambda} \leq \frac{R-f}{c}, \end{cases} \tag{EC.207}$$

where $q_d^* = (\mu - c/(R-f))\,\lambda^{-1}$ is the unique solution of $1/(\mu-\lambda q_d^*) = (R-f)/c$, and the equilibrium average sojourn time is

$$\widehat{W}_d = \begin{cases} \frac{R-f}{c} & \text{if } \rho<1 \text{ and } \frac{R-f}{c} < \frac{1}{\mu-\lambda}, \text{ or } \rho\geq 1 \\ \frac{1}{\mu-\lambda} & \text{if } \rho<1 \text{ and } \frac{1}{\mu-\lambda} \leq \frac{R-f}{c}. \end{cases}$$

(Note that we did not include the condition $1/\mu < (R-f)/c$ in the first line of (EC.207) because (EC.195) already implies that.) From Little's law, we have $\widehat{L}_d = \widehat{\lambda}_{e,d}\widehat{W}_d$. Therefore,

$$\widehat{L}_d = \begin{cases} \frac{(R-f)\mu}{c} - 1 & \text{if } \rho<1 \text{ and } \frac{R-f}{c} < \frac{1}{\mu-\lambda}, \text{ or } \rho\geq 1 \\ \frac{\lambda}{\mu-\lambda} & \text{if } \rho<1 \text{ and } \frac{1}{\mu-\lambda} \leq \frac{R-f}{c}. \end{cases}$$

Plugging the expressions above in $\widehat{SW}_d$ formula (EC.194), we complete the proof of Lemma EC.12. □

K.2. Proof of Proposition 5-(a)

Recall Remark EC.1. Recall also Lemmas EC.11 and EC.12, and their proofs. Note that the unobservable pooled system is an M/M/N system with the equilibrium effective arrival rate (EC.204) and each unobservable dedicated subsystem (that consists of one dedicated line and its server) is an M/M/1 system with the equilibrium effective arrival rate (EC.207). Note also that $\widetilde{W}_p(Nx)$ represents the average sojourn time in the M/M/N system with the total effective arrival rate Nx and the service rate μ for each server, and $\widetilde{W}_d(x)$ represents the average sojourn time in the M/M/1 system with the effective arrival rate x and service rate μ.

We claim and show in Lemma EC.13 at the end of this section that

$$\widetilde{W}_p(Nx) \le \widetilde{W}_d(x) \qquad \text{for } x < \mu, \tag{EC.208}$$

and hence

$$\widetilde{W}_p(N\lambda) \le \widetilde{W}_d(\lambda) \qquad \text{for } \rho < 1. \tag{EC.209}$$

Using (EC.209), we will prove the claim in Proposition 5-(a) under two main cases. Case 1: Suppose that $\rho < 1$ and $\frac{1}{\mu-\lambda} \le \frac{R-f}{c}$. Then, by Lemma EC.12 and its proof, $\widehat{W}_d = \frac{1}{\mu-\lambda}$, $\widehat{q}_d = 1$ and $\widehat{\lambda}_{e,d} = \lambda$ in equilibrium. This and (EC.209) imply that $\widetilde{W}_p(N\lambda) \le \frac{R-f}{c}$. Then, by the proof of Lemma EC.11, $\widehat{q}_p = 1$ and $\widehat{\lambda}_{e,p} = N\lambda$ in equilibrium. As a result,

$$\widehat{W}_p = \widetilde{W}_p(N\lambda) \le \widetilde{W}_d(\lambda) = \widehat{W}_d. \tag{EC.210}$$

Recall the social welfare from (EC.194), and recall that $\widehat{\lambda}_{e,d}N = \widehat{\lambda}_{e,p} = N\lambda$. Then,

$$\widehat{SW}_d = (R - c\widehat{W}_d)\widehat{\lambda}_{e,d}N = (R - c\widehat{W}_d)\lambda N \quad \text{and} \quad \widehat{SW}_p = (R - c\widehat{W}_p)\widehat{\lambda}_{e,p} = (R - c\widehat{W}_p)\lambda N.$$

Because $\widehat{W}_p \le \widehat{W}_d$ by (EC.210), $\widehat{SW}_p \ge \widehat{SW}_d$. This completes the proof of Proposition 5-(a) under Case 1.

Case 2: Suppose now that either $\rho < 1$ and $\frac{R-f}{c} < \frac{1}{\mu-\lambda}$, or $\rho \ge 1$. From Lemma EC.12, it follows that, in equilibrium, the average sojourn time in the unobservable dedicated system is

$$\widehat{W}_d = \frac{R-f}{c}, \tag{EC.211}$$

and the equilibrium social welfare in the dedicated system is

$$\widehat{SW}_d = (R - c\widehat{W}_d)\widehat{\lambda}_{e,d}N = f\widehat{\lambda}_{e,d}N. \tag{EC.212}$$

Given these performance metrics in the dedicated system, we now prove the claim by considering the following two subcases for $\widetilde{W}_p(N\lambda)$.

Case 2.1: Suppose that $\widetilde{W}_p(N\lambda) \ge \frac{R-f}{c}$. Then, by Lemma EC.11, the equilibrium average sojourn time in the pooled system is $\widehat{W}_p = \frac{R-f}{c}$, which is equal to $\widehat{W}_d$ by (EC.211). Thus, by Lemma EC.11,

$$\widehat{SW}_p = (R - c\widehat{W}_p)\widehat{\lambda}_{e,p} = f\widehat{\lambda}_{e,p}. \tag{EC.213}$$

We now show that

$$\widehat{\lambda}_{e,p} \ge \widehat{\lambda}_{e,d}N. \tag{EC.214}$$

Suppose for a contradiction that $\widehat{\lambda}_{e,p} < N\widehat{\lambda}_{e,d}$. Then, (EC.208) and the fact that $\widetilde{W}_p(x)$ strictly increases in x for $x < N\mu$ imply that

$$\widehat{W}_p = \widetilde{W}_p(\widehat{\lambda}_{e,p}) < \widetilde{W}_p(N\widehat{\lambda}_{e,d}) \leq \widetilde{W}_d(\widehat{\lambda}_{e,d}) = \widehat{W}_d. \tag{EC.215}$$

But, this contradicts with $\widehat{W}_p = \widehat{W}_d$. Thus, we have (EC.214). Based on (EC.214), from (EC.212) and (EC.213), it follows that $\widehat{SW}_p \geq \widehat{SW}_d$.

Case 2.2: Suppose that $\widetilde{W}_p(N\lambda) < \frac{R-f}{c}$. Then, $\widehat{q}_p = 1$ and $\widehat{\lambda}_{e,p} = N\lambda$ in equilibrium. Thus, $R - f - c\widehat{W}_p = R - f - c\widetilde{W}_p(N\lambda) > 0$, which implies that the equilibrium long-run average sojourn time in the pooled system satisfies $\widehat{W}_p < \frac{R-f}{c} = \widehat{W}_d$. Thus, the equilibrium social welfare in the pooled system is

$$\widehat{SW}_p = (R - c\widehat{W}_p)\widehat{\lambda}_{e,p} = (R - c\widehat{W}_p)N\lambda > (R - c\widehat{W}_d)N\lambda \geq (R - c\widehat{W}_d)\widehat{\lambda}_{e,d}N = \widehat{SW}_d,$$

which completes the proof of Case 2.2.

Note that combining Case 1 and Case 2 covers the entire parameter space. Thus, the claim in Proposition 5-(a) follows.

The following lemma shows our claim in (EC.208).

LEMMA EC.13. *$\widetilde{W}_p(Nx) \leq \widetilde{W}_d(x)$ for $x/\mu < 1$.*

Proof of Lemma EC.13: Suppose that $x/\mu < 1$. Recall that $\widetilde{W}_p(Nx)$ represents the average sojourn time in the M/M/N system with the total effective arrival rate Nx and the service rate μ for each server, and $\widetilde{W}_d(x)$ represents the average sojourn time in the M/M/1 system with the effective arrival rate x and service rate μ. Denote by X_d total number of customers in N of the M/M/1 lines in the steady-state, and let X_p be the corresponding figure in the aforementioned M/M/N system. Based on this, to show our claim, we will use standard likelihood comparison technique (see, for instance, Smith and Whitt (1981)). Let $\theta_p(m+1)$ be the transition rate from state $m+1$ to m in the pooled system (in the steady-state), $\theta_d(m+1|S_t)$ be the transition rate from state $m+1$ to m in the dedicated system and S_t is the state of the dedicated system (i.e., number of customers in each of the N lines) at time t, for any $m = 0, 1, \ldots$. Because $\theta_d(m+1|S_t) \leq \theta_p(m+1)$ for each m regardless of S_t, in the steady-state, we have

$$\mathbb{P}(X_d = m)Nx \leq \mathbb{P}(X_d = m+1)\theta_p(m+1) \quad \text{and} \quad \mathbb{P}(X_p = m)Nx = \mathbb{P}(X_p = m+1)\theta_p(m+1).$$

Thus, we have

$$\frac{\mathbb{P}(X_d = m+1)}{\mathbb{P}(X_d = m)} \geq \frac{Nx}{\theta_p(m+1)} = \frac{\mathbb{P}(X_p = m+1)}{\mathbb{P}(X_p = m)}. \tag{EC.216}$$

Using this, we now show that $\mathbb{E}(X_d) \geq \mathbb{E}(X_p)$. Note that (EC.216) implies that $\frac{\mathbb{P}(X_d=j)}{\mathbb{P}(X_d=i)} \geq \frac{\mathbb{P}(X_p=j)}{\mathbb{P}(X_p=i)}$ for all $i \leq j$, $i, j \in \mathbb{N}$, which is equivalent to

$$\mathbb{P}(X_d = j)\mathbb{P}(X_p = i) \geq \mathbb{P}(X_d = i)\mathbb{P}(X_p = j). \tag{EC.217}$$

The summation on both sides of (EC.217) over i from 0 to j gives

$$\mathbb{P}(X_d = j)\mathbb{P}(X_p \leq j) \geq \mathbb{P}(X_d \leq j)\mathbb{P}(X_p = j). \tag{EC.218}$$

Similarly, the summation on both sides of (EC.217) over j from $i+1$ to ∞ results in

$$\mathbb{P}(X_d \geq i+1)\mathbb{P}(X_p = i) \geq \mathbb{P}(X_d = i)\mathbb{P}(X_p \geq i+1). \tag{EC.219}$$

Combining (EC.218) and (EC.219) and letting $i = j = a$, we have

$$\frac{\mathbb{P}(X_d \geq a+1)}{\mathbb{P}(X_p \geq a+1)} \geq \frac{\mathbb{P}(X_d = a)}{\mathbb{P}(X_p = a)} \geq \frac{\mathbb{P}(X_d \leq a)}{\mathbb{P}(X_p \leq a)}. \tag{EC.220}$$

Thus, $\mathbb{P}(X_d \leq a) \leq \mathbb{P}(X_p \leq a)$ for any non-negative integer a, and hence

$$\mathbb{E}(X_d) = \sum_{i=0}^{\infty}(1 - \mathbb{P}(X_d \leq i)) \geq \sum_{i=0}^{\infty}(1 - \mathbb{P}(X_p \leq i)) = \mathbb{E}(X_p). \tag{EC.221}$$

Observe that the long-run average number of customers in one of the N separate dedicated sub-systems (i.e., $\widetilde{L}_d$) and the long-run average number of customers in the pooled system (i.e., $\widetilde{L}_p$) satisfy $\widetilde{L}_d = \mathbb{E}(X_d)/N$ and $\widetilde{L}_p \doteq \mathbb{E}(X_p)$. Then, by Little's Law and (EC.221),

$$\widetilde{W}_d(x) = \frac{\widetilde{L}_d(x)}{x} = \frac{\mathbb{E}(X_d)/N}{x} = \frac{\mathbb{E}(X_d)}{Nx} \geq \frac{\mathbb{E}(X_p)}{Nx} = \frac{\widetilde{L}_p(Nx)}{Nx} = \widetilde{W}_p(Nx).$$

This completes the proof of the claim. □ □

K.3. Proof of Proposition 5-(b)

Let us first formalize the formulations mentioned in the statement of Proposition 5-(b).

(i) *Welfare-maximizing fee*: For each system $j \in \{d,p\}$, a fee f_j is chosen to maximize the social welfare:

$$\max_{f_j \geq 0} \widehat{SW}_j \doteq (R - c\widehat{W}_j(f_j))N\lambda\widehat{q}_j(f_j), \quad j \in \{d,p\}. \tag{EC.222}$$

(ii) *Revenue-maximizing fee*: For each system $j \in \{d,p\}$, a fee f_j is chosen to maximize the service provider's revenue:

$$\max_{f_j \geq 0} \widehat{RV}_j \doteq N\lambda\widehat{q}_j(f_j)f_j, \quad j \in \{d,p\}. \tag{EC.223}$$

In both (EC.222) and (EC.223), $\widehat{q}_j(f_j)$ represents the equilibrium joining probability for $j \in \{d,p\}$.

We first prove the statement in the absence of parentheses. Recall that the maximum social welfare under the welfare maximization formulation is as in (EC.222). Note that (EC.222), i.e., choosing the fee to maximize equilibrium social welfare, is equivalent to choosing the effective arrival rate to maximize equilibrium social welfare as below:

$$\widehat{SW}_j^* = \begin{cases} \max_{0 \leq x \leq \lambda}(R - c\widetilde{W}_j(x))xN & \text{if } j = d, \\ \max_{0 \leq x \leq \lambda}(R - c\widetilde{W}_j(Nx))xN & \text{if } j = p. \end{cases} \tag{EC.224}$$

We only need to focus on the case that $x < \mu$ in both systems since the the system will be unstable otherwise. We already know that $\widetilde{W}_p(Nx) \leq \widetilde{W}_d(x)$ for $x/\mu < 1$ according to Lemma EC.13. Thus, $\widehat{SW}_p^* \geq \widehat{SW}_d^*$.

We now prove the statement in the parentheses. Note that (EC.223) is equivalent to choosing the effective arrival rate to maximize the equilibrium revenue

$$\widehat{RV}_j^{**} = \begin{cases} \max_{0 \leq x \leq \lambda}(R - c\widetilde{W}_j(x))xN & \text{if } j = d, \\ \max_{0 \leq x \leq \lambda}(R - c\widetilde{W}_j(Nx))xN & \text{if } j = p. \end{cases} \tag{EC.225}$$

This and (EC.224) imply that the maximum revenue is the same as the maximum social welfare in equilibrium. Then, the claim in parentheses immediately follows from Proposition 5-(b) in the absence of parentheses. □

Appendix L: Proof of Proposition 6

L.1. Proof of Part (a)

We already proved in Proposition 5-(a) that the unobservable pooled system outperforms the unobservable dedicated system in social welfare. Based on this, we only need to compare the unobservable pooled system with the observable pooled system to prove Proposition 6-(a). Below, we will show that the observable pooled system results in larger equilibrium social welfare than the unobservable pooled system. There can be two cases related to $\widetilde{W}_p(N\lambda)$:

Case A: Suppose that $\widetilde{W}_p(N\lambda) > \frac{R}{c}$. Then, by Lemma EC.11, in the unobservable pooled system, average sojourn time and social welfare in equilibrium are the following, respectively:

$$\widehat{W}_p = \frac{R}{c} \quad \text{and} \quad \widehat{SW}_p = (R - c\widehat{W}_p)\widehat{\lambda}_{e,p} = 0. \tag{EC.226}$$

Recall the social welfare SW_p in the observable pooled system from Lemma 1. We claim and show below that $SW_p > 0$. Combining this with (EC.226), we have $\widehat{SW}_p < SW_p$.

It only remains to prove our claim that $SW_p > 0$. Recall from (EC.195) that $\frac{R\mu}{c} > 1$. Then, each joining customer receives a non-negative expected net benefit by joining. Furthermore, a customer that finds $n \le N-1$ customers in the system upon arrival receives strictly positive expected net benefit $R - \bar{W}_p(n+1)c = R - \frac{1}{\mu}c > 0$ for $\frac{R\mu}{c} > 1$. As a result, $SW_p > 0$ in this case.

Case B: Suppose that $\widetilde{W}_p(N\lambda) \le \frac{R}{c}$. Then, by Lemma EC.11 and its proof, $\widehat{q}_p = 1$ and $\widehat{\lambda}_{e,p} = N\lambda$ for the unobservable pooled system. We now show that the observable pooled system results in strictly larger social welfare than the unobservable pooled system, i.e., $SW_p > \widehat{SW}_p$. Recall the SW_p from Lemma 1, and observe that using the stationary probability distribution $\{\pi_0, \pi_1, \ldots, \pi_K\}$ for the number of customers in the observable pooled system (in the proof of Lemma 1), SW_p can also be expressed as follows:

$$\begin{aligned}
SW_p =& N\lambda \left(\sum_{i=0}^{N-1} \left(R - \frac{c}{\mu}\right) \frac{N^i}{i!}\rho^i + \sum_{i=N}^{K-1} \left(R - \frac{(i+1)c}{N\mu}\right) \frac{N^N}{N!}\rho^i \right) \Big/ \left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \sum_{i=N}^{K} \frac{N^N}{N!}\rho^i \right) \\
=& N\lambda \left(\sum_{i=0}^{N-1} (R - \bar{W}_p(i+1)c) \frac{N^i}{i!}\rho^i + \sum_{i=N}^{K-1} (R - \bar{W}_p(i+1)c) \frac{N^N}{N!}\rho^i \right) \Big/ \left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \sum_{i=N}^{K} \frac{N^N}{N!}\rho^i \right) \\
>& N\lambda \left(\sum_{i=0}^{N-1} (R - \bar{W}_p(i+1)c) \frac{N^i}{i!}\rho^i + \sum_{i=N}^{\infty} (R - \bar{W}_p(i+1)c) \frac{N^N}{N!}\rho^i \right) \Big/ \left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \sum_{i=N}^{\infty} \frac{N^N}{N!}\rho^i \right) \quad \text{(EC.227)} \\
=& N\lambda R - cN\lambda \left(\sum_{i=0}^{N-1} \bar{W}_p(i+1) \frac{N^i}{i!}\rho^i + \sum_{i=N}^{\infty} \bar{W}_p(i+1) \frac{N^N}{N!}\rho^i \right) \Big/ \left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \sum_{i=N}^{\infty} \frac{N^N}{N!}\rho^i \right) \\
=& N\lambda R - cN\lambda \left(\sum_{i=0}^{N-1} \frac{1}{\mu} \frac{N^i}{i!}\rho^i + \sum_{i=N}^{\infty} \frac{i+1}{N\mu} \frac{N^N}{N!}\rho^i \right) \Big/ \left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \sum_{i=N}^{\infty} \frac{N^N}{N!}\rho^i \right) \\
=& N\lambda R - c \left(\sum_{i=0}^{N-1} \frac{N^{i+1}}{i!}\rho^{i+1} + \sum_{i=N}^{\infty} \frac{N^N}{N!}(i+1)\rho^{i+1} \right) \Big/ \left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \sum_{i=N}^{\infty} \frac{N^N}{N!}\rho^i \right) \\
=& N\lambda R - c \left(\sum_{i=0}^{N-1} \frac{N^i}{i!} i\rho^i + \sum_{i=N}^{\infty} \frac{N^N}{N!} i\rho^i \right) \Big/ \left(\sum_{i=0}^{N-1} \frac{N^i}{i!}\rho^i + \sum_{i=N}^{\infty} \frac{N^N}{N!}\rho^i \right) \quad \text{(EC.228)} \\
=& \widehat{SW}_p
\end{aligned}$$

Here, the inequality (EC.227) holds because $R - \bar{W}_p(i+1)c = R - \frac{i+1}{N\mu}c < 0$ for any $i \ge K \doteq \lfloor \frac{RN\mu}{c} \rfloor$.

Combining Cases A and B, it follows that the observable pooled systems results in larger social welfare than the unobservable pooled system. This and Proposition 5-(a) complete our proof for Proposition 6-(a). □

L.2. Proof of Part (b)

According to Propositions 4 and 5-(b), when the service fee is set to maximize the social welfare, the maximum social welfare in the pooled system is larger than that in the dedicated system for both observable and unobservable cases. Thus, to prove the claim, we only need to compare the observable pooled system with the unobservable pooled system. To do so, consider an alternative setting in which admissions to an M/M/N system can be controlled rather than customers making their own joining/balking decisions. Among all admission control policies (including the randomized ones), the optimal admission rule that maximizes the social welfare is a deterministic control limit rule that induces a queue capacity. (This is because this alternative formulation corresponds to a finite state Markov decision process.) Because that optimal queue capacity can be achieved by imposing a service fee in the observable pooled system where customers make their own joining decisions, the observable pooled system achieves larger maximum social welfare than the unobservable pooled system when in each system, the fee is set to maximize the social welfare. From this, the claim immediately follows. □

Appendix M: Statement and Proof of Proposition EC.2

PROPOSITION EC.2. *Suppose that the customer service benefit is distributed with a general distribution function $G(\cdot)$ defined on any bounded support. Then, in equilibrium, the social welfare (average sojourn time) under the unobservable pooled system is greater (smaller) than or equal to the one under the unobservable dedicated system.*

Proof of Proposition EC.2: Suppose that the service benefit R has a general distribution with the p.d.f. $g(\cdot)$ and the c.d.f $G(\cdot)$ defined on the support $[L,H]$, and consider unobservable systems. We prove the claim under each of the three possible cases about $c\widetilde{W}_d(\cdot)$ below:

Case 1: Suppose that $c\widetilde{W}_d(\lambda)\le L$. Then, $\lambda<\mu$ must be true and in equilibrium all customers join the dedicated system. By Lemma EC.13, $c\widetilde{W}_p(N\lambda)\le c\widetilde{W}_d(\lambda)\le L$, thus all customers in the pooled system also join. As a result, in equilibrium, $\widehat{W}_p=\widetilde{W}_p(N\lambda)\le\widetilde{W}_d(\lambda)=\widehat{W}_d$ and $\widehat{SW}_p=N\lambda\int_L^H g(x)(x-c\widehat{W}_p)dx\ge N\lambda\int_L^H g(x)(x-c\widehat{W}_d)dx=\widehat{SW}_d$.

Case 2: Suppose that $c\widetilde{W}_d(0)=\frac{c}{\mu}>H$. Then, no customer in the dedicated system joins. Moreover, no customer joins in the pooled system as well since $c\widetilde{W}_p(0)=\frac{c}{\mu}>H$.

Case 3: Suppose that $c\widetilde{W}_d(\lambda)>L$ and $c\widetilde{W}_d(0)\le H$. Then, there exists a threshold benefit a_d^e such that only customers with benefit larger than a_d^e join in equilibrium and $a_d^e=c\widetilde{W}_d(\lambda\bar{G}(a_d^e))$ where $\bar{G}(x)\doteq 1-G(x)$. Note that a_d^e is the intersection point of functions $y_1(x)\doteq x$ and $y_2(x)\doteq c\widetilde{W}_d(\lambda\bar{G}(x))$ for $x\in[L,H]$. (The intersection point exists and is unique as $y_1(\cdot)$ is strictly increasing and $y_2(\cdot)$ is strictly decreasing.) We consider two possible subcases: Case 3.1: Suppose that $c\widetilde{W}_p(N\lambda)\le L$. Then, in the pooled system, all customers join. Because $c\widetilde{W}_p(N\lambda)\le L<a_d^e=c\widetilde{W}_d(\lambda\bar{G}(a_d^e))$, $\widetilde{W}_p(N\lambda)<\widetilde{W}_d(\lambda\bar{G}(a_d^e))$, i.e., $\widehat{W}_p<\widehat{W}_d$, and $\widehat{SW}_p=N\lambda\int_L^H g(x)(x-c\widehat{W}_p)dx>N\lambda\int_{a_d^e}^H g(x)(x-c\widehat{W}_p)dx>N\lambda\int_{a_d^e}^H g(x)(x-c\widehat{W}_d)dx=\widehat{SW}_d$. Case 3.2: Suppose that $c\widetilde{W}_p(N\lambda)>L$ and $c\widetilde{W}_p(0)\le H$. Then, there exits a threshold benefit a_p^e such that only customers with benefit larger than a_p^e join in equilibrium and $a_p^e=c\widetilde{W}_p(N\lambda\bar{G}(a_p^e))$. Note that a_p^e is the intersection point of functions $y_1(x)$ and $y_3(x)\doteq c\widetilde{W}_p(N\lambda\bar{G}(x))$ for $x\in[L,H]$. Since $y_3(x)=c\widetilde{W}_p(N\lambda\bar{G}(x))\le c\widetilde{W}_d(\lambda\bar{G}(x))=y_2(x)$ for any $x\in[L,H]$, $a_p^e\le a_d^e$ and $\widetilde{W}_p(N\lambda\bar{G}(a_p^e))\le\widetilde{W}_d(\lambda\bar{G}(a_d^e))$, i.e., $\widehat{W}_p\le\widehat{W}_d$. As a result, $\widehat{SW}_p=N\lambda\int_{a_p^e}^H g(x)(x-c\widehat{W}_p)dx\ge N\lambda\int_{a_d^e}^H g(x)(x-c\widehat{W}_p)dx\ge N\lambda\int_{a_d^e}^H g(x)(x-c\widehat{W}_d)dx=\widehat{SW}_d$. □

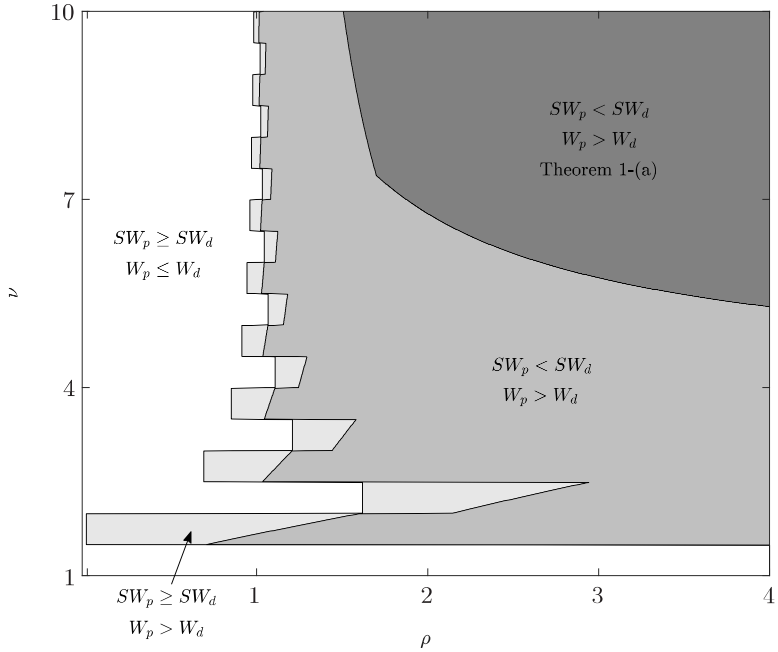

Figure EC.1 The comparison of pooled and dedicated systems with the following parameters: $c = 1$, $\mu = 1$ and $N = 2$.

Appendix N: Additional Numerical Studies for Sections 3 and 5

This section includes Figures EC.1 and EC.2 that provide additional numerical examples for Sections 3 and 5.

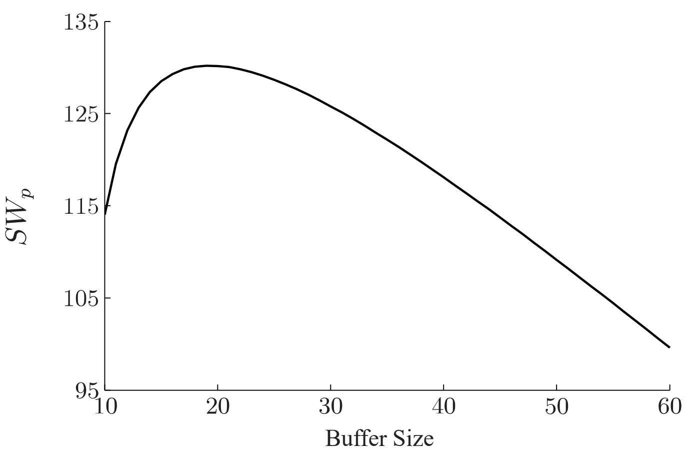

Figure EC.2 The following parameters are used: $R = 15$, $c = 1$, $N = 10$, $\mu = 1$ and $\lambda = 1.1$.