Online Appendix

Appendix A: Proof of Theorem 1.

The proof relies on several theorems from Sennott (2009). First, according to Theorem 7.2.3 of Sennott (2009), if the (SEN) Assumptions as stated in the reference hold, then we know that there exists a finite constant J and a finite function h that satisfy the ACOI (average cost optimal inequalities)

$$g + h(i) \geq \min_a \left\{ C(i,a) + \sum_j P_{ij}(a)h(j) \right\}, \, i \in \mathcal{S},$$

where

$$C(0) = 0;\, C(r,n) = C(p,n) = nc_w,\, \forall n \geq 1,$$

and for $1 \leq i \leq K$ and $n \geq 1$,

$$C(\alpha_i, n; 0) = nc_w + \alpha_i \gamma_S c_{ad},\, C(\alpha_i, n; 1) = nc_w + \gamma_B c_{tr}.$$

Furthermore, there exists an average cost optimal policy f that achieves the minimum in the ACOI.

Now, according to Theorem 7.5.6 of Sennott (2009), if the (BOR) Assumptions as stated in the reference, hold then we know that the (SEN) Assumptions also hold. Hence we only need to show that (BOR) Assumptions hold under the condition that

$$\lambda \left(\frac{1}{\gamma_S} + \frac{\alpha}{\gamma_B} \right) < 1.$$

We consider the stationary policy d that always chooses to request a hospital bed at the time of triage. Then under the assumption that $\lambda \left(\frac{1}{\gamma_S} + \frac{\alpha}{\gamma_B} \right) < 1$, the Markov chain induced by d is an M/G/1 queue with service time, denoted by L, which can be described as

$$L = \begin{cases} S & \text{w.p. } 1-\alpha \\ \max(S,B) & \text{w.p. } \alpha \end{cases},$$

thus

$$\begin{aligned} \mathrm{E}[L] &= (1-\alpha)\,\mathrm{E}[S] + \alpha \mathrm{E}[\max(S,B)] \\ &\leq (1-\alpha)\,\mathrm{E}[S] + \alpha \mathrm{E}[S+B] \\ &= (1-\alpha)\,\frac{1}{\gamma_S} + \alpha \left(\frac{1}{\gamma_S} + \frac{1}{\gamma_B} \right) \\ &= \frac{1}{\gamma_S} + \frac{\alpha}{\gamma_B}, \end{aligned}$$

then,

$$\rho = \lambda \mathrm{E}[L] \leq \lambda \left(\frac{1}{\gamma_S} + \frac{\alpha}{\gamma_B} \right) < 1,$$

and hence the Markov chain under d is a positive recurrent class $\mathcal{R}_d = \mathbb{Y} = \{0\} \cup \{(m,n) \mid m \in \{\alpha_i\}_{i=1}^M \cup \{r,p\}, 1 \leq n \leq l\}$.

For any $z \in \mathcal{R}_d = \mathbb{Y}$, d is a z-standard policy because the Markov chain under d is positive recurrent (see Definitions 7.5.1 and C.2.5 in Sennott (2009)), and hence the expected first passage time and associated total expected cost from any one state to another are both finite. Thus, condition (BOR1) holds.

Now, since the resulting Markov chain under d is positive recurrent, the long run average cost under d, denoted by J_d, is finite. Choose $\varepsilon = 1$. Define $D = \{s \mid C(s,a) \leq J_d + 1 \text{ for some } a\}$. Then, $D = \{0\} \cup A \cup B$, where

$$A = \{(\alpha_i, n) \mid 1 \leq i \leq M \text{ and } 1 \leq n \leq \left\lfloor \frac{1}{c_w} \left(J_d + 1 - \min\{\alpha_i \gamma_S c_{ad}, \gamma_B c_{tr}\}\right) \right\rfloor \},$$

and

$$B = \{(r,n) \text{ and } (p,n) \mid 1 \leq n \leq \left\lfloor \frac{J_d + 1}{c_w} \right\rfloor \}.$$

It is easy to see that D is a finite set since J_d is finite and thus we can conclude that (BOR2) holds.

Finally, (BOR3) holds because $D \subset \mathcal{R}_d$, and thus $D - \mathcal{R}_d = \emptyset$. This completes the proof of the theorem. □

Appendix B: Proof of Theorem 2.

First, we introduce the finite-horizon version of the uniformized, discrete-time version of our problem described in Section 5.3. Let $V_m^\pi(x)$ denote the total expected cost under policy π over a period of m stages starting from state x. The optimal expected m-stage cost then can be expressed as

$$V_m(x) = \inf_{\pi \in \Pi} V_m^\pi(x),$$

and satisfies the following finite horizon optimality equations: For $m \geq 1$,

$$V_m(0) = \lambda \sum_j q_j V_{m-1}(\alpha_j, 1) + (\gamma_S + \gamma_B) V_{m-1}(0). \tag{EC.B.1}$$

For all $m \geq 1$, $n \geq 1$, and $\alpha_i \in \Omega$,

$$\begin{aligned} V_m(\alpha_i, n) = {} & n c_w + \lambda V_{m-1}(\alpha_i, n+1) + (1-\alpha_i)\gamma_S \sum_j q_j V_{m-1}(\alpha_j, n-1) \\ & + \alpha_i \gamma_S V_{m-1}(p,n) + \gamma_B \min\{V_{m-1}(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_{m-1}(r,n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\}, \end{aligned} \tag{EC.B.2}$$

where $V_m(\alpha_j, 0) = V_m(0) = \sum_j q_j V_m(\alpha_j, 0)$, for all j.

For all $m \geq 1$ and $n \geq 1$,

$$V_m(r,n) = n c_w + \lambda V_{m-1}(r, n+1) + \gamma_S \sum_j q_j V_{m-1}(\alpha_j, n-1) + \gamma_B V_{m-1}(r,n), \tag{EC.B.3}$$

$$V_m(p,n) = nc_w + \lambda V_{m-1}(p,n+1) + \gamma_B \sum_j q_j V_{m-1}(\alpha_j, n-1) + \gamma_S V_{m-1}(p,n). \qquad \text{(EC.B.4)}$$

Next, we show that the optimality operator preserves certain conditions as stated in the following lemma. It is important to note that while only some of the conditions stated in the lemma will be key to establishing the threshold result, the proof of those essential conditions requires showing all of them together.

B.1. Lemmas needed for the proof of Theorem 2

LEMMA EC.1. *Suppose for any $m \geq 1$ we have that*

1) $V_m(r,n) - \sum_j q_j V_m(\alpha_j, n-1)$ is a non-negative non-decreasing function of n for all $n \geq 1$.

2) $V_m(\alpha_i,n) - \sum_j q_j V_m(\alpha_j, n-1)$ is a non-decreasing function of n for all i and $n \geq 1$ and $V_m(\alpha_i, 1) - \sum_j q_j V_m(\alpha_j, 0) \geq \alpha_i c_{ad}$.

3) $\min\{V_m(\alpha_i,n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r,n) + \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr}\} - V_m(r,n)$ is a non-decreasing function of n for all i and $n \geq 1$.

Then we have

Condition 1. $\min\{V_m(\alpha_i,n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r,n) + \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr}\} - \sum_j q_j V_m(\alpha_i, n-1)$ *is a non-decreasing function of n for all i and $n \geq 1$, and* $\min\{V_m(\alpha_i,1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r,1) + \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr}\} - \sum_j q_j V_m(\alpha_i, 0) \geq \frac{\alpha_i(\gamma_S+\gamma_B)}{\gamma_B} c_{ad}$ *for all i.*

Proof of Lemma EC.1. For the first part of Condition 1, we first write

$$\begin{aligned}
&\min\{V_m(\alpha_i,n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r,n) + \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr}\} - \sum_j q_j V_m(\alpha_j, n-1) \\
&\qquad = \left[\min\{V_m(\alpha_i,n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r,n) + \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr}\} - V_m(r,n)\right] \\
&\qquad\qquad + \left[V_m(r,n) - \sum_j q_j V_m(\alpha_j, n-1)\right].
\end{aligned}$$

Then, the first part of the condition immediately follows by noting that the right hand side is a non-decreasing function of n for all $\alpha_i \in \Omega$ and $n \geq 1$ using 3) and 1) for V_m.

To show the second part of Condition 1, let $N_m(\alpha_i) = \inf\{n : V_m(\alpha_i,n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} > V_m(r,n) + \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr}\}$. First suppose that $N_m(\alpha_i) = 1$. Then

$$\begin{aligned}
&\min\{V_m(\alpha_i,1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r,1) + \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr}\} - \sum_j q_j V_m(\alpha_j, 0) \\
&\qquad = \left[V_m(r,1) - \sum_j q_j V_m(\alpha_j, 0)\right] + \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr} \\
&\qquad\qquad \geq \frac{\gamma_S+\gamma_B}{\gamma_B} c_{tr} \geq \frac{\alpha_i(\gamma_S+\gamma_B)}{\gamma_B} c_{ad},
\end{aligned}$$

where for the first inequality, we used 1) for V_m. Now, suppose that $N_m(\alpha_i) \geq 2$. Then

$$\begin{aligned}\min\{V_m(\alpha_i,1)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\} - \sum_j q_j V_m(\alpha_j,0)& \\ =\left[V_m(\alpha_i,1)-\sum_j q_j V_m(\alpha_j,0)\right]+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}& \\ \geq \alpha_i c_{ad}+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}=\frac{\alpha_i(\gamma_S+\gamma_B)}{\gamma_B}c_{ad},&\end{aligned}$$

where the inequality follows from 2) for V_m. This completes the proof for the condition. □

LEMMA EC.2. *Suppose for any $m \geq 1$ we have that $V_m(\alpha_i,n)-V_m(r,n)$ is a non-decreasing function of n for all i and $n \geq 1$. Then we have*

Condition 2. $V_m(\alpha_i,n)-\min\{V_m(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}$ *is a non-decreasing function of n for all i and $n \geq 1$, and* $V_m(\alpha_i,1)-\min\{V_m(\alpha_i,1)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\} \geq -\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}$ *for all i.*

Condition 3. $\min\{V_m(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}-V_m(r,n)$ *is a non-decreasing function of n for all i and $n \geq 1$.*

Proof of Lemma EC.2. Proof of Condition 2: The second part of the condition is immediate by noting that

$$\begin{aligned}V_m(\alpha_i,n)-\min\{V_m(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\} \geq V_m(\alpha_i,n)-\left[V_m(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}\right]& \\ =-\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}.&\end{aligned}$$

To prove the first part of the condition, we need to show that for all i and $n \geq 1$

$$\begin{aligned}V_m(\alpha_i,n+1)-\min\{V_m(\alpha_i,n+1)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,n+1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}& \\ \geq V_m(\alpha_i,n)-\min\{V_m(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}.&\end{aligned}$$

Now, let $N_m(\alpha_i)=\inf\{n : V_m(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad} > V_m(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}$. Then, since $V_m(\alpha_i,n)-V_m(r,n)$ is a non-decreasing function of n for all i and $n \geq 1$ we have that $V_m(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad} > V_m(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}$ if and only if $n \geq N_m(\alpha_i)$. Consequently, for $1 \leq n \leq N_m(\alpha_i)-2$ we have

$$\begin{aligned}\left[V_m(\alpha_i,n+1)-\min\{V_m(\alpha_i,n+1)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,n+1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right]& \\ -\left[V_m(\alpha_i,n)-\min\{V_m(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_m(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right]=& \\ \left[V_m(\alpha_i,n+1)-V_m(\alpha_i,n+1)-\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}\right]& \\ -\left[V_m(\alpha_i,n)-V_m(\alpha_i,n)-\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}\right]=0\geq 0.&\end{aligned}$$

For $n \geq N_m(\alpha_i)$ we have

$$\left[V_m(\alpha_i, n+1) - \min\{V_m(\alpha_i, n+1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, n+1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\}\right] - \left[V_m(\alpha_i, n) - \min\{V_m(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\}\right] = \left[V_m(\alpha_i, n+1) - V_m(r, n+1) - \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\right] - \left[V_m(\alpha_i, n) - V_m(r, n) - \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\right] = [V_m(\alpha_i, n+1) - V_m(r, n+1)] - [V_m(\alpha_i, n) - V_m(r, n)] \geq 0,$$

because $V_m(\alpha_i, n) - V_m(r, n)$ is a non-decreasing function of n by the assumption of the lemma. For $n = N_m(\alpha_i) - 1$ we have

$$\left[\min\{V_m(\alpha_i, N_m(\alpha_i)) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, N_m(\alpha_i)) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, N_m(\alpha_i))\right] - \left[\min\{V_m(\alpha_i, N_m(\alpha_i) - 1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, N_m(\alpha_i) - 1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, N_m(\alpha_i) - 1)\right] = \left[V_m(r, N_m(\alpha_i)) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr} - V_m(r, N_m(\alpha_i))\right] - \left[V_m(\alpha_i, N_m(\alpha_i) - 1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} - V_m(r, N_m(\alpha_i) - 1)\right] = \left[V_m(r, N_m(\alpha_i) - 1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\right] - \left[V_m(\alpha_i, N_m(\alpha_i) - 1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}\right] \geq 0,$$

which follows from the definition of $N_m(\alpha_i)$. Thus we have proved Condition 2.

<u>Proof of Condition 3:</u> We need to show that for all $n \geq 1$

$$\min\{V_m(\alpha_i, n+1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, n+1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, n+1) \geq \min\{V_m(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, n).$$

Again, let $N_m(\alpha_i)$ be defined as before when establishing Condition 2. For $1 \leq n \leq N_m(\alpha_i) - 2$ we have

$$\left[\min\{V_m(\alpha_i, n+1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, n+1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, n+1)\right] - \left[\min\{V_m(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, n)\right] = \left[V_m(\alpha_i, n+1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} - V_m(r, n+1)\right] - \left[V_m(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} - V_m(r, n)\right] = [V_m(\alpha_i, n+1) - V_m(r, n+1)] - [V_m(\alpha_i, n) - V_m(r, n)] \geq 0,$$

because by assumption, $V_m(\alpha_i, n) - V_m(r, n)$ is non-decreasing with respect to n.

For $n \geq N_m(\alpha_i)$ we have

$$\left[\min\{V_m(\alpha_i, n+1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, n+1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, n+1)\right] - \left[\min\{V_m(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, n)\right] = \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr} - \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr} = 0 \geq 0.$$

When $n = N_m(\alpha_i) - 1$ we have

$$\begin{aligned}
&\left[\min\{V_m(\alpha_i, N_m(\alpha_i)) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, N_m(\alpha_i)) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, N_m(\alpha_i))\right] \\
&- \left[\min\{V_m(\alpha_i, N_m(\alpha_i) - 1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_m(r, N_m(\alpha_i) - 1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_m(r, N_m(\alpha_i) - 1)\right] \\
&= \left[V_m(r, N_m(\alpha_i)) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr} - V_m(r, N_m(\alpha_i))\right] \\
&- \left[V_m(\alpha_i, N_m(\alpha_i) - 1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} - V_m(r, N_m(\alpha_i) - 1)\right] \\
&= \left[V_m(r, N_m(\alpha_i) - 1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\right] - \left[V_m(\alpha_i, N_m(\alpha_i) - 1) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}\right] \geq 0,
\end{aligned}$$

which follows from the definition of $N_m(\alpha_i)$. Thus we have established Condition 3. □

LEMMA EC.3. *Let $\alpha_i c_{ad} \leq c_{tr}$, for all $\alpha_i \in \Omega$ and suppose that the following six conditions all hold for $0 \leq k \leq m - 1$ where $m \geq 1$:*

Condition 4. $V_k(\alpha_i, n) - V_k(r, n)$ *is a non-decreasing function of* n *for all* i *and* $n \geq 1$.

Condition 5. $V_k(p, n) - \sum_j q_j V_k(\alpha_j, n-1)$ *is a non-negative non-decreasing function of* n *for all* $n \geq 1$.

Condition 6. $V_k(\alpha_i, n) - (1 - \alpha_i) \sum_j q_j V_k(\alpha_j, n-1) - \alpha_i V_k(p, n)$ *is a non-decreasing function of* n *for all* i *and* $n \geq 1$, *and* $V_k(\alpha_i, 1) - (1 - \alpha_i) \sum_j q_j V_k(\alpha_j, 0) - \alpha_i V_k(p, 1) \geq \alpha_i c_{ad}$.

Condition 7. $V_k(r, n) - \sum_j q_j V_k(\alpha_j, n-1)$ *is a non-negative non-decreasing function of* n *for all* $n \geq 1$.

Condition 8. $V_k(\alpha_i, n) - \sum_j q_j V_k(\alpha_j, n-1)$ *is a non-decreasing function of* n *for all* i *and* $n \geq 1$ *and* $V_k(\alpha_i, 1) - \sum_j q_j V_k(\alpha_j, 0) \geq \alpha_i c_{ad}$.

Condition 9. $V_k(\alpha_i, n)$ *is a non-decreasing function of* i *for all* $n \geq 1$.

Then Condition 4 through 9 also hold for $k = m$, i.e., Condition 4 through 9 are preserved under the optimality equations.

Proof of Lemma EC.3. To show each of the properties we use induction in a similar manner. Let $m \geq 1$ and suppose that condition 4 through 9 as given in the statement of the lemma all hold for $0 \leq k \leq m - 1$. We will show that the same conditions then also hold for $k = m$.

First, using Lemma EC.2 and Condition 4 with $0 \le k \le m-1$ we can conclude that Condition 2 and 3 also hold for all $0 \le k \le m-1$. Then, using Lemma EC.1 and noting that the three conditions stated in the lemma are satisfied by Condition 7, 8, and 3 for $0 \le k \le m-1$, we can conclude that Condition 1 also holds for all $0 \le k \le m-1$.

Proof of Condition 4: For $n \ge 1$, using (EC.B.2) and (EC.B.3) we have

$$\begin{aligned}
&V_m(\alpha_i, n) - V_m(r, n) = \\
&\qquad \lambda \left[V_{m-1}(\alpha_i, n+1) - V_{m-1}(r, n+1)\right] + \alpha_i \gamma_S \left[V_{m-1}(p, n) - \sum_j q_j V_{m-1}(\alpha_j, n-1)\right] \\
&\qquad\qquad + \gamma_B \left[\min\{V_{m-1}(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad}, V_{m-1}(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\} - V_{m-1}(r, n)\right].
\end{aligned}$$

From Conditions 4, 5 and 3 (with $k = m-1$), we know that the right hand side of the above equation is non-decreasing in n for $n \ge 1$ and thus we can conclude that Condition 4 also holds for $k = m$.

Proof of Condition 5: For $n \ge 2$, using (EC.B.4) and (EC.B.2) we have

$$\begin{aligned}
&V_m(p, n) - \sum_j q_j V_m(\alpha_j, n-1) \\
&\quad = c_w + \lambda \left[V_{m-1}(p, n+1) - \sum_j q_j V_{m-1}(\alpha_j, n)\right] + \gamma_S \left[V_{m-1}(p, n) - \sum_j q_j V_{m-1}(\alpha_j, n-1)\right] \\
&\qquad + \gamma_S \sum_j q_j \left[V_{m-1}(\alpha_j, n-1) - (1-\alpha_j) \sum_k q_k V_{m-1}(\alpha_k, n-2) - \alpha_j V_{m-1}(p, n-1)\right] \\
&\quad + \gamma_B \sum_j q_j \left[V_{m-1}(\alpha_j, n-1) - \min\{V_{m-1}(\alpha_j, n-1) + \frac{\alpha_j \gamma_S}{\gamma_B} c_{ad}, V_{m-1}(r, n-1) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\}\right].
\end{aligned}$$

From Conditions 5, 6 and 2 (with $k = m-1$), we know that the right hand side of the above equation is non-decreasing in n for $n \ge 2$ and thus we can conclude that Condition 5 also holds for $k = m$ but when $n \ge 2$. To establish the condition for the case of $n = 1$, we need to show that $V_m(p, 1) - \sum_j q_j V_m(\alpha_j, 0) \ge 0$ and $V_m(p, 2) - \sum_j q_j V_m(\alpha_j, 1) \ge V_m(p, 1) - \sum_j q_j V_m(\alpha_j, 0)$. To establish the first inequality, using (EC.B.4) and the fact that $\lambda + \gamma_S + \gamma_B = 1$, we can write

$$\begin{aligned}
&V_m(p, 1) - \sum_j q_j V_m(\alpha_j, 0) \\
&\qquad = c_w + \lambda \left[V_{m-1}(p, 2) - \sum_j q_j V_{m-1}(\alpha_j, 1)\right] + \gamma_S \left[V_{m-1}(p, 1) - \sum_j q_j V_{m-1}(\alpha_j, 0)\right],
\end{aligned}$$

which is non-negative by Condition 5 (with $k = m-1$).

To establish the second inequality, we can write

$$\begin{aligned}
&\left[V_m(p,2)-\sum_j q_j V_m(\alpha_j,1)\right]-\left[V_m(p,1)-\sum_j q_j V_m(\alpha_j,0)\right]= \\
&\quad\lambda\left\{\left[V_{m-1}(p,3)-\sum_j q_j V(\alpha_j,2)\right]-\left[V_{m-1}(p,2)-\sum_j q_j V(\alpha_j,1)\right]\right\} \\
&\quad+\gamma_S\left\{\left[V_{m-1}(p,2)-\sum_j q_j V_{m-1}(\alpha_j,1)\right]-\left[V_{m-1}(p,1)-\sum_j q_j V_{m-1}(\alpha_j,0)\right]\right\} \\
&\quad+\gamma_S\sum_j q_j\left[V_{m-1}(\alpha_j,1)-(1-\alpha_j)\sum_k q_k V_{m-1}(\alpha_k,0)-\alpha_j V_{m-1}(p,1)\right] \\
&\quad+\gamma_B\sum_j q_j\left[V_{m-1}(\alpha_j,1)-\min\{V_{m-1}(\alpha_j,1)+\frac{\alpha_j\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right]\geq \\
&\quad\gamma_S\sum_j q_j\left[V_{m-1}(\alpha_j,1)-(1-\alpha_j)\sum_k q_k V_{m-1}(\alpha_k,0)-\alpha_j V_{m-1}(p,1)\right] \\
&\quad+\gamma_B\sum_j q_j\left[V_{m-1}(\alpha_j,1)-\min\{V_{m-1}(\alpha_j,1)+\frac{\alpha_j\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right]\geq \\
&\quad\gamma_S\sum_j q_j\alpha_j c_{ad}-\gamma_B\sum_j q_j\frac{\alpha_j\gamma_S}{\gamma_B}c_{ad}=0,
\end{aligned}$$

where we used Conditions 5, 6 and 2 (with $k=m-1$). Hence, Condition 5 also holds for $k=m$.

Proof of Condition 6: For $n\geq 2$, using (EC.B.2) and (EC.B.4) we have

$$\begin{aligned}
&V_m(\alpha_i,n)-(1-\alpha_i)\sum_j q_j V_m(\alpha_j,n-1)-\alpha_i V_m(p,n)=(1-\alpha_i)c_w \\
&\quad+\lambda\left[V_{m-1}(\alpha_i,n+1)-(1-\alpha_i)\sum_j q_j V_{m-1}(\alpha_j,n)-\alpha_i V_{m-1}(p,n+1)\right] \\
&\quad+\gamma_B\left[\min\{V_{m-1}(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}-\sum_j q_j V_{m-1}(\alpha_j,n-1)\right] \\
&\quad+(1-\alpha_i)\gamma_S\sum_j q_j\left[V_{m-1}(\alpha_j,n-1)-(1-\alpha_j)\sum_k q_k V_{m-1}(\alpha_k,n-2)-\alpha_j V_{m-1}(p,n-1)\right] \\
&\quad+(1-\alpha_i)\gamma_B\sum_j q_j\left[V_{m-1}(\alpha_j,n-1)-\min\{V_{m-1}(\alpha_j,n-1)+\frac{\alpha_j\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,n-1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right].
\end{aligned}$$

From Conditions 6, 1 and 2 (with $k=m-1$), we know that the right hand side of the above equation is non-decreasing in n for $n\geq 2$ and thus we can conclude that the first part of Condition 6 also holds for $k=m$ but when $n\geq 2$. To complete the proof for Condition 6, it then remains to show that $V_m(\alpha_i,1)-(1-\alpha_i)\sum_j q_j V_m(\alpha_j,0)-\alpha_i V_m(p,1)\geq \alpha_i c_{ad}$ and $V_m(\alpha_i,2)-(1-\alpha_i)\sum_j q_j V_m(\alpha_j,1)-\alpha_i V_m(p,2)\geq V_m(\alpha_i,1)-(1-\alpha_i)\sum_j q_j V_m(\alpha_j,0)-\alpha_i V_m(p,1)$.

To establish the first inequality, first, using (EC.B.2) (with $n=1$) we have

$$V_m(\alpha_i,1)=c_w+\lambda V_{m-1}(\alpha_i,2)+(1-\alpha_i)\gamma_S\sum_j q_jV_{m-1}(\alpha_j,0)$$
$$+\alpha_i\gamma_S V_{m-1}(p,1)+\gamma_B\min\{V_{m-1}(\alpha_i,1)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\},$$

and from (EC.B.1) we have

$$(1-\alpha_i)\sum_j q_jV_m(\alpha_j,0)=(1-\alpha_i)V_m(0)=(1-\alpha_i)\lambda\sum_j q_jV_{m-1}(\alpha_j,1)+(1-\alpha_i)(\gamma_S+\gamma_B)\sum_j q_jV_{m-1}(\alpha_j,0).$$

Using (EC.B.4) (with $n=1$) we can write

$$\alpha_iV_m(p,1)=\alpha_ic_w+\alpha_i\lambda V_{m-1}(p,2)+\alpha_i\gamma_B\sum_j q_jV_{m-1}(\alpha_j,0)+\alpha_i\gamma_S V_{m-1}(p,1).$$

It then follows that

$$V_m(\alpha_i,1)-(1-\alpha_i)\sum_j q_jV_m(\alpha_j,0)-\alpha_iV_m(p,1)=(1-\alpha_i)c_w$$
$$+\lambda\left[V_{m-1}(\alpha_i,2)-(1-\alpha_i)\sum_j q_jV_{m-1}(\alpha_j,1)-\alpha_iV_{m-1}(p,2)\right]$$
$$+\gamma_B\left[\min\{V_{m-1}(\alpha_i,1)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}-\sum_j q_jV_{m-1}(\alpha_j,0)\right].$$

Then, using Condition 6 and 1 (with $k=m-1$), we have

$$V_m(\alpha_i,1)-(1-\alpha_i)\sum_j q_jV_m(\alpha_j,0)-\alpha_iV_m(p,1)\geq\lambda\alpha_ic_{ad}+\gamma_B\cdot\frac{\gamma_S+\gamma_B}{\gamma_B}\cdot\alpha_ic_{ad}=\alpha_ic_{ad}.$$

For the second inequality, we can write

$$\left[V_m(\alpha_i,2)-(1-\alpha_i)\sum_j q_jV_m(\alpha_j,1)-\alpha_iV_m(p,2)\right]-\left[V_m(\alpha_i,1)-(1-\alpha_i)\sum_j q_jV_m(\alpha_j,0)-\alpha_iV_m(p,1)\right]=$$
$$\lambda\left\{\left[V_{m-1}(\alpha_i,3)-(1-\alpha_i)\sum_j q_jV_{m-1}(\alpha_j,2)-\alpha_iV_{m-1}(p,3)\right]\right.$$
$$\left.-\left[V_{m-1}(\alpha_i,2)-(1-\alpha_i)\sum_j q_jV_{m-1}(\alpha_j,1)-\alpha_iV_{m-1}(p,2)\right]\right\}$$
$$+\gamma_B\left\{\left[\min\{V_{m-1}(\alpha_i,2)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,2)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}-\sum_j q_jV_{m-1}(\alpha_j,1)\right]\right.$$
$$\left.-\left[\min\{V_{m-1}(\alpha_i,1)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}-\sum_j q_jV_{m-1}(\alpha_j,0)\right]\right\}$$

$$+(1-\alpha_i)\gamma_S\sum_j q_j\left[V_{m-1}(\alpha_j,1)-(1-\alpha_j)\sum_k q_k V_{m-1}(\alpha_k,0)-\alpha_j V_{m-1}(p,1)\right]$$
$$+(1-\alpha_i)\gamma_B\sum_j q_j\left[V_{m-1}(\alpha_j,1)-\min\{V_{m-1}(\alpha_j,1)+\frac{\alpha_j\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right]\geq$$
$$+(1-\alpha_i)\gamma_S\sum_j q_j\left[V_{m-1}(\alpha_j,1)-(1-\alpha_j)\sum_k q_k V_{m-1}(\alpha_k,0)-\alpha_j V_{m-1}(p,1)\right]$$
$$+(1-\alpha_i)\gamma_B\sum_j q_j\left[V_{m-1}(\alpha_j,1)-\min\{V_{m-1}(\alpha_j,1)+\frac{\alpha_j\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right]\geq$$
$$(1-\alpha_i)\left[\gamma_S\sum_j q_j\alpha_j c_{ad}-\gamma_B\sum_j q_j\frac{\alpha_j\gamma_S}{\gamma_B}c_{ad}\right]=0,$$

where we used Condition 6, 1 and 2 for $k=m-1$. Thus Condition 6 for $k=m$ follows.

Proof of Condition 7: For $n\geq 2$, we have

$$V_m(r,n)-\sum_j q_j V_m(\alpha_j,n-1)=$$
$$c_w+\lambda\left[V_{m-1}(r,n+1)-\sum_j q_j V_{m-1}(\alpha_j,n)\right]$$
$$+\gamma_B\left[V_{m-1}(r,n)-\sum_j q_j V_{m-1}(\alpha_j,n-1)\right]$$
$$+\gamma_S\sum_j q_j\left[V_{m-1}(\alpha_j,n-1)-(1-\alpha_j)\sum_k q_k V_{m-1}(\alpha_k,n-2)-\alpha_j V_{m-1}(p,n-1)\right]$$
$$+\gamma_B\sum_j q_j\left[V_{m-1}(\alpha_j,n-1)-\min\{V_{m-1}(\alpha_j,n-1)+\frac{\alpha_j\gamma_S c_{ad}}{\gamma_B},V_{m-1}(r,n-1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right].$$

Then, using Condition 7, 6 and 2 (with $k=m-1$), we can conclude that the right hand side of the above equation is non-decreasing in n for $n\geq 2$ and thus $V_m(r,n)-\sum_j q_j V_m(\alpha_j,n-1)$ is also non-decreasing in n for $n\geq 2$. To complete the proof for Condition 7, we need to show that $V_m(r,1)-\sum_j q_j V_m(\alpha_j,0)\geq 0$ and $V_m(r,2)-\sum_j q_j V_m(\alpha_j,1)\geq V_m(r,1)-\sum_j q_j V_m(\alpha_j,0)$.

To establish the first inequality, first, using (EC.B.3) with $n=1$, we can write

$$V_m(r,1)=c_w+\lambda V_{m-1}(r,2)+\gamma_S\sum_j q_j V_{m-1}(\alpha_j,0)+\gamma_B V_{m-1}(r,1),$$

and using (EC.B.1), we can write

$$\sum_j q_j V_m(\alpha_j,0)=V_m(0)=\lambda\sum_j q_j V_{m-1}(\alpha_j,1)+(\gamma_S+\gamma_B)\sum_j q_j V_{m-1}(\alpha_j,0),$$

where we used the fact that $\sum_j q_j V_k(\alpha_j,0)=V_k(0),\forall k\geq 0$. Thus, we have

$$V_m(r,1)-\sum_j q_j V_m(\alpha_j,0)=c_w+\lambda\left[V_{m-1}(r,2)-\sum_j q_j V_{m-1}(\alpha_j,1)\right]+\gamma_B\left[V_{m-1}(r,1)-\sum_j q_j V_{m-1}(\alpha_j,0)\right].$$

From Condition 7 (with $k = m-1$), we can then see that $V_m(r,1) - \sum_j q_j V_m(\alpha_j, 0) \geq 0$.

To establish the second inequality, we can write

$$\left[V_m(r,2) - \sum_j q_j V_m(\alpha_j,1)\right] - \left[V_m(r,1) - \sum_j q_j V_m(\alpha_j,0)\right] =$$

$$\lambda\left\{\left[V_{m-1}(r,3) - \sum_j q_j V_{m-1}(\alpha_j,2)\right] - \left[V_{m-1}(r,2) - \sum_j q_j V_{m-1}(\alpha_j,1)\right]\right\}$$
$$+\gamma_B\left\{\left[V_{m-1}(r,2) - \sum_j q_j V_{m-1}(\alpha_j,1)\right] - \left[V_{m-1}(r,1) - \sum_j q_j V_{m-1}(\alpha_j,0)\right]\right\}$$
$$+\gamma_S \sum_j q_j \left[V_{m-1}(\alpha_j,1) - (1-\alpha_j)\sum_k q_k V_{m-1}(\alpha_k,0) - \alpha_j V_{m-1}(p,1)\right]$$
$$+\gamma_B \sum_j q_j \left[V_{m-1}(\alpha_j,1) - \min\{V_{m-1}(\alpha_j,1) + \frac{\alpha_j\gamma_S c_{ad}}{\gamma_B}, V_{m-1}(r,1) + \frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right] \geq$$
$$\gamma_S \sum_j q_j \left[V_{m-1}(\alpha_j,1) - (1-\alpha_j)\sum_k q_k V_{m-1}(\alpha_k,0) - \alpha_j V_{m-1}(p,1)\right]$$
$$+\gamma_B \sum_j q_j \left[V_{m-1}(\alpha_j,1) - \min\{V_{m-1}(\alpha_j,1) + \frac{\alpha_j\gamma_S c_{ad}}{\gamma_B}, V_{m-1}(r,1) + \frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}\right] \geq$$
$$\gamma_S \sum_j q_j \alpha_j c_{ad} - \gamma_B \sum_j q_j \frac{\alpha_j \gamma_S}{\gamma_B} c_{ad} = 0,$$

where we used Conditions 7, 6 and 2 (with $k = m-1$). Thus Condition 7 for $k = m$ follows.

<u>Proof of Condition 8:</u> First, we can write

$$\left[V_m(\alpha_i,n+1) - \sum_j q_j V_m(\alpha_j,n)\right] - \left[V_m(\alpha_i,n) - \sum_j q_j V_m(\alpha_j,n-1)\right] =$$
$$\{[V_m(\alpha_i,n+1) - V_m(r,n+1)] - [V_m(\alpha_i,n) - V_m(r,n)]\}$$
$$+\left\{\left[V_m(r,n+1) - \sum_j q_j V_m(\alpha_j,n)\right] - \left[V_m(r,n) - \sum_j q_j V_m(\alpha_j,n-1)\right]\right\}.$$

Then, using Conditions 4 and 7 (with $k = m$), which we have already established, we can conclude that for all $n \geq 1$, $V_m(\alpha_i,n+1) - \sum_j q_j V_m(\alpha_j,n) \geq V_m(\alpha_i,n) - \sum_j q_j V_m(\alpha_j,n-1)$. It then remains to show that $V_m(\alpha_i,1) - \sum_j q_j V_m(\alpha_j,0) \geq \alpha_i c_{ad}$.

Using (EC.B.2) (with $n = 1$), we have

$$V_m(\alpha_i,1) = c_w + \lambda V_{m-1}(\alpha_i,2) + (1-\alpha_i)\gamma_S \sum_j q_j V_{m-1}(\alpha_i,0)$$
$$+\alpha_i\gamma_S V_{m-1}(p,1) + \gamma_B \min\{V_{m-1}(\alpha_i,1) + \frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, V_{m-1}(r,1) + \frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\},$$

and using (EC.B.1), we have

$$\sum_j q_j V_m(\alpha_j,0) = V_m(0) = \lambda \sum_j q_j V_{m-1}(\alpha_j,1) + (\gamma_S+\gamma_B)\sum_j q_j V_{m-1}(\alpha_j,0),$$

where we used the fact that $\sum_j q_j V_k(\alpha_j,0)=V_k(0)$, $\forall k\geq 0$. Then, we have

$$V_m(\alpha_i,1)-\sum_j q_jV_m(\alpha_j,0)=c_w+\lambda\left[V_{m-1}(\alpha_i,2)-\sum_j q_jV_{m-1}(\alpha_j,1)\right]+\alpha_i\gamma_S\left[V_{m-1}(p,1)-\sum_j q_jV_{m-1}(\alpha_j,0)\right]$$
$$+\gamma_B\left[\min\{V_{m-1}(\alpha_i,1)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad},V_{m-1}(r,1)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}-\sum_j q_jV_{m-1}(\alpha_j,0)\right].$$

Hence

$$V_m(\alpha_i,1)-\sum_j q_jV_m(\alpha_j,0)\geq\lambda\alpha_ic_{ad}+0+\gamma_B\cdot\frac{\alpha_i(\gamma_S+\gamma_B)}{\gamma_B}c_{ad}=\lambda\alpha_ic_{ad}+(\gamma_S+\gamma_B)\alpha_ic_{ad}=\alpha_ic_{ad},$$

where we used Conditions 8, 5 and 1 for $k=m-1$. Thus, Condition 8 for $k=m$ follows.

<u>Proof of Condition 9:</u> Using (EC.B.2), we have, for $n\geq 1$

$$V_m(\alpha_i,n)=nc_w+\lambda V_{m-1}(\alpha_i,n+1)+\gamma_S\sum_j q_jV_{m-1}(\alpha_j,n-1)+$$
$$\alpha_i\gamma_S\left[V_{m-1}(p,n)-\sum_j q_jV_{m-1}(\alpha_j,n-1)\right]$$
$$+\gamma_B\min\{V_{m-1}(\alpha_i,n)+\frac{\alpha_i\gamma_Sc_{ad}}{\gamma_B},V_{m-1}(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}\}.$$

The first and the third term are both invariant in i. The second term is non-decreasing in i using Condition 9 (with $k=m-1$). The fourth term is non-decreasing in i because α_i is non-decreasing in i and $V_{m-1}(p,n)-\sum_j q_jV_{m-1}(\alpha_j,n-1)$ is invariant in i and is non-negative according to Condition 5 (with $k=m-1$). The last term is also non-decreasing in i from the fact that minimization preserves monotonicity and that Condition 9 holds for $k=m-1$. Thus, we can conclude that $V_m(\alpha_i,n)$ is non-decreasing in i for all $n\geq 1$. This completes the proof of the lemma. □

B.2. Main body of the proof of Theorem 2

Now, we choose the terminating costs so that $V_0(\alpha_i,n)=nc_{ad}$ for $n\geq 0$ and $\alpha_i\in\Omega$, $V_0(r,n)=V_0(p,n)=(n-1)c_{ad}$ for $n\geq 1$. One can then easily check that all the conditions of Lemma EC.3 hold for $k=0$. Then, repeated use of Lemma EC.3 implies that all the conditions of the lemma hold for any integer $m\geq 1$. We also know from Theorem 1 that there exists an optimal policy for the long-run average cost problem with bias function $h(\cdot)$ satisfying the ACOEs (5.2) through (5.5). Thus, we must have

$$h(\alpha_i,n)-h(r,n)=\lim_{m\to\infty}[V_m(\alpha_i,n)-V_m(r,n)],$$

for $\alpha_i\in\Omega$ and $n\geq 1$. Then, because we know that all the conditions of Lemma EC.3 hold for any m and in particular Condition 1, i.e., $V_m(\alpha_i,n)-V_m(r,n)$ is a non-decreasing function of n, we can conclude that $h(\alpha_i,n)-h(r,n)$ is also non-decreasing in n for $n\geq 1$ and $\alpha_i\in\Omega$.

Let $N(\alpha_i)$ be as defined in the theorem. It is easy to see that for all $n \geq N(\alpha_i)$, we have

$$h(\alpha_i, n) + \frac{\alpha_i \gamma_S c_{ad}}{\gamma_B} > h(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr},$$

i.e., it is optimal to call ahead for a bed at the time of triage. This completes the proof of Theorem 2. □

Appendix C: Proof of Theorem 3.

The proof follows along the lines of the proof of Theorem 2. First, note that we only need to prove that $h(\alpha_i, n) - h(r, n)$, or simply, $h(\alpha_i, n)$, is non-decreasing in i for all $n \geq 1$. Now, choose the terminating costs so that $V_0(\alpha_i, n) = nc_{ad}$ for $n \geq 0$ and $\alpha_i \in \Omega$, $V_0(r, n) = V_0(p, n) = (n-1)c_{ad}$ for $n \geq 1$. One can then easily check that the conditions of Lemma EC.3 hold for $k = 0$. Then, repeated use of Lemma EC.3 implies that all the conditions of the lemma hold for any integer $m \geq 1$. We also know from Theorem 1 that there exists an optimal policy for the long-run average cost problem with bias function $h(\cdot)$ satisfying the ACOEs (5.2) through (5.5). Thus, we must have

$$h(\alpha_i, n) - h(r, n) = \lim_{m \to \infty} [V_m(\alpha_i, n) - V_m(r, n)]$$

for $\alpha_i \in \Omega$ and $n \geq 1$. Then, because we know that the conditions of Lemma EC.3 hold for any m and in particular Condition 9, i.e., $V_m(\alpha_i, n) - V_m(r, n)$ is a non-decreasing function of α_i, we can conclude that $h(\alpha_i, n) - h(r, n)$ is also non-decreasing in α_i. This completes the proof of the theorem. □

Appendix D: Proof of Theorem 4.

We only need to show that $V(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} \geq V(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}$ if and only if $n \geq \frac{(c_{tr} - \alpha_i c_{ad})(\gamma_S + \gamma_B)}{\alpha_i c_w}$. Subtracting (3.8) from (3.7) we get

$$\begin{aligned} V(\alpha_i, n) - V(r, n) = &\frac{\alpha_i \gamma_S}{\gamma_S + \gamma_B} \left[V(p, n) - \sum_j q_j V(\alpha_j, n-1) \right] \\ &+ I_{\{V(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} \geq V(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\}} c_{tr} \\ &+ I_{\{V(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} < V(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\}} \frac{\gamma_B}{\gamma_S + \gamma_B} \left[V(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} - V(r, n) \right]. \end{aligned}$$

Rearranging (3.9) we have

$$V(p, n) - \sum_j q_j V(\alpha_j, n-1) = \frac{nc_w}{\gamma_B},$$

and then we can write

$$\begin{aligned} V(\alpha_i, n) - V(r, n) = &\frac{\alpha_i \gamma_S n c_w}{\gamma_B(\gamma_S + \gamma_B)} \\ &+ I_{\{V(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} \geq V(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\}} c_{tr} \\ &+ I_{\{V(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} < V(r, n) + \frac{\gamma_S + \gamma_B}{\gamma_B} c_{tr}\}} \frac{\gamma_B}{\gamma_S + \gamma_B} \left[V(\alpha_i, n) + \frac{\alpha_i \gamma_S}{\gamma_B} c_{ad} - V(r, n) \right]. \quad \text{(EC.D.1)} \end{aligned}$$

Hence, if $V(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}\geq V(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}$, or, $V(\alpha_i,n)-V(r,n)\geq \frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}-\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}$, then we have

$$\begin{aligned} V(\alpha_i,n)-V(r,n)=\frac{\alpha_i\gamma_S n c_w}{\gamma_B(\gamma_S+\gamma_B)}+c_{tr}&\geq \frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}-\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}\\ \implies \frac{\alpha_i\gamma_S n c_w}{\gamma_B(\gamma_S+\gamma_B)}&\geq \frac{\gamma_S}{\gamma_B}c_{tr}-\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}\\ \implies n&\geq \frac{(c_{tr}-\alpha_i c_{ad})(\gamma_S+\gamma_B)}{\alpha_i c_w}. \end{aligned}$$

Now we are only left to show that if $n\geq \frac{(c_{tr}-\alpha_i c_{ad})(\gamma_S+\gamma_B)}{\alpha_i c_w}$, then $V(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}\geq V(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}$. Suppose not, i.e., suppose we have $V(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}< V(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}$, then using (EC.D.1), we have

$$\begin{aligned} V(\alpha_i,n)-V(r,n)&=\frac{\alpha_i\gamma_S n c_w}{\gamma_B(\gamma_S+\gamma_B)}+\frac{\gamma_B}{\gamma_S+\gamma_B}\left[V(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}-V(r,n)\right]\\ &\geq \frac{\alpha_i\gamma_S c_w}{\gamma_B(\gamma_S+\gamma_B)}\cdot\frac{(c_{tr}-\alpha_i c_{ad})(\gamma_S+\gamma_B)}{\alpha_i c_w}+\frac{\gamma_B}{\gamma_S+\gamma_B}\left[V(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}-V(r,n)\right]\\ &=\frac{\gamma_S(c_{tr}-\alpha_i c_{ad})}{\gamma_B}+\frac{\gamma_B}{\gamma_S+\gamma_B}\left[V(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}-V(r,n)\right]\\ &=\frac{\gamma_B}{\gamma_S+\gamma_B}\left[V(\alpha_i,n)-V(r,n)\right]+\frac{\gamma_S(c_{tr}-\alpha_i c_{ad})}{\gamma_B}+\frac{\alpha_i\gamma_S}{\gamma_S+\gamma_B}c_{ad}\\ &=\frac{\gamma_B}{\gamma_S+\gamma_B}\left[V(\alpha_i,n)-V(r,n)\right]+\frac{\gamma_S}{\gamma_B}\left[c_{tr}-\frac{\alpha_i\gamma_S}{\gamma_S+\gamma_B}c_{ad}\right]\\ \implies \frac{\gamma_S}{\gamma_S+\gamma_B}\left[V(\alpha_i,n)-V(r,n)\right]&\geq \frac{\gamma_S}{\gamma_B}\left[c_{tr}-\frac{\alpha_i\gamma_S}{\gamma_S+\gamma_B}c_{ad}\right]\\ \implies V(\alpha_i,n)-V(r,n)&\geq \frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}-\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}, \end{aligned}$$

which is a contradiction. Hence our assumption that $V(\alpha_i,n)+\frac{\alpha_i\gamma_S}{\gamma_B}c_{ad}< V(r,n)+\frac{\gamma_S+\gamma_B}{\gamma_B}c_{tr}$ does not hold. This completes the proof of the theorem. □

Appendix E: The Simulation Model

Currently, at the ED where our data came from, no early bed requests are made for the patients. (This was also the case during 2012, the year our data were collected.) Therefore, we first developed a simulation model that is meant to capture the system, as it was during the period the data were collected, with no early bed requests, and validated it. Then, we developed a version of the simulation model that allows the possibility of making early bed requests using either the policies we propose or one of the benchmarks. In the rest of this section, we first describe the model for the ED. Then, we explain how we validated the model. Finally, we describe how we incorporated early bed requests. Note that the simulation models were built and run using the discrete-event simulation software Arena version 15.

E.1. Description of the Base Simulation Model

Each patient visiting the ED follows these main steps: the patient arrives, joins the queue for triage, and goes through triage when it is his/her turn. After triage, the patient either is admitted to the ED right away or waits in the waiting room until a bed for the patient becomes available. Once the patient is admitted to the ED, s/he goes through two stages. The first stage is what we refer to as the *ED workup* during which the patient is seen by the nurses and physicians as many times as needed and any procedures (such as blood test, CT scan etc.) that are needed to diagnose the problem are performed. At the end of this first stage, a disposition decision is made for the patient and the second stage starts. This second stage either corresponds to *discharge time* or *boarding time* for the patient. If the decision is to discharge the patient from the ED, the patient discharge procedure is activated and at the completion of this procedure the patient vacates the bed and leaves the ED. If the decision is to admit the patient to the hospital then the second stage corresponds to the patient's boarding time. Boarding time of a patient depends on the availability of the hospital beds and the TPP time for the patient. Specifically, when the admit decision is made if a hospital bed is available then the boarding time simply equals to the TPP time; however, if there are no beds available, then the hospital bed request joins a queue and waits for a bed to become available. (As patients are discharged from the hospital and beds become available, they are allocated to the requests in this queue in a FCFS manner.) In this case, the boarding time of a patient is equal to the waiting time in the queue plus the TPP time. Once the second stage service is over the patient leaves the ED making the bed she occupied available for patients who are waiting or will arrive in the future. Patients who are transferred to the hospital keep a single bed occupied during their stay. Once their hospital stay is over, they leave and the bed they occupied become available for admitted ED patients who are waiting for a hospital bed or future ED patients who will be admitted to the hospital. We should note that modelling of boarding times (waiting for a hospital bed availability plus an allocation delay) is based on the approach developed by Shi et al. (2016). Unlike Shi et al. (2016), however, we consider only a single allocation delay, which is experienced once a hospital bed is allocated to the admitted patient. Considering two types of allocation delays, one prior to bed assignment and one followed by bed assignment, captures the actual system in a more realistic way; however, for our purposes, assuming a single allocation delay is a more reasonable choice since we do not have data that can be used to reliably estimate the two delays separately and we use the mean allocation delay as one of the calibration parameters for validating the simulation model. For our purposes, having a single allocation delay, which can be seen as pre and post-allocation delays (as described in Shi et al. (2016)) put together, captures the actual system realistically and helps us avoid adding unnecessary complexity to the simulation model.

Our initial analysis revealed that the arrival rates of the patients, their hospital admission probabilities, their ED workup, boarding, and discharge times depended on day-of-the-week, time-of-day, whether or not the patient is an adult or pediatric patient, the patient's ESI level, and whether the patient is eventually admitted to the hospital or discharged from the ED. (Obviously, whether or not a patient is going to be eventually admitted is not an observable feature and is not known when making decisions but the simulation model knows this for each patient when generating the times the patient spends in different stages etc.) With 5 different ESI levels, two age levels (adult vs. pediatric), and two different possible eventual disposition decisions (admit vs. discharge), we put patients into one of $5 \times 2 \times 2 = 20$ classes.

Patient Arrivals: In line with most prior work, we found that a non-homogeneous Poisson process would be a good fit for arrival times to the ED. Specifically, we used the method developed in Brown et al. (2005) to test the hypothesis that patient arrivals follow a non-homogeneous Poisson process. More specifically, to ameliorate the effects of rounding errors in data collection, we added a uniform noise to each data point and we divided the data into 672 subgroups based on hour of day, day of week, and season of year, to ensure some level of stationarity in each subgroup. For each subgroup, we performed a Kolmogorov-Smirnov (K-S) test and found that the null hypothesis was rejected (at a significance level of 0.01) for only 25 out of 672 subgroups. Based on this statistical analysis, in our simulation model, we generated arrivals for each class using an independent non-homogeneous Poisson process. The arrival rates for each process are described as step functions of time with each step corresponding to a specific one-hour window of every week and thus resulting in a total of $24 \times 7 = 168$ steps. However, note that patient arrival rate functions corresponding to certain days were very similar to each other giving us no strong justification for using a different function for each day. Therefore, in the simulation study, we used three different arrival rate functions with one for *Mondays only*, another for *Tuesday through Friday*, and another for *Saturday and Sunday.* Tables EC.7 through EC.14 in Appendix E.4 list the arrival rates estimated and assumed in the simulation model for each patient class.

Triage, waiting room, and bed assignment: There are two triage nurses working at all times. The main goal of triage is to determine how critical each patient's condition is and assign an ESI level to the patient. If an arriving patient finds both triage nurses busy, she joins the queue for triage, which progresses in a first-come-first-served fashion. Unfortunately, our data did not contain elements that would allow us to fit a probability distribution or estimate the mean for triage times and therefore using common practice, we assumed triage times to have a triangular distribution and we used the best estimates by practitioners for the parameters of the distribution. Specifically, we assumed triangular distribution with a support from 5 minutes to 15 minutes and a peak at 10 minutes. (Because triage times are relatively much shorter than the times spent in the ED,

Table EC.1 Patient Prioritization and Pod Assignment Rules During the Year 2012.

Age group	**Acuity class**	**Priority**	**Assigned pod** (in order of pref. as capacity allows)
Adult	ESI1	1	Trauma beds, Pod A, Pod B
Adult	ESI2	2	Pod A, Pod B
Adult	ESI3 (Acute)	3	Pod A, Pod B
Adult	ESI3 (Non-acute)	4	Pod D, Pod A, Pod B
Adult	ESI4 and ESI5	4	Pod D when open, otherwise, Pod A, Pod B
Pediatric	ESI1	1	Ped. pod, Pod A, Pod B
Pediatric	ESI2	2	Ped. pod, Pod A, Pod B
Pediatric	ESI3	4	Ped. pod, Pod A, Pod B
Pediatric	ESI4 and ESI5	4	Ped. pod when open, otherwise, Pod A, Pod B

Note: Roughly 50% of ESI3 patients are acute.

estimation errors, unless they are extremely off, are unlikely to alter the main conclusions.) After triage, if there is a bed available for the patient, the patient is admitted to the ED right away. If not, the patient starts waiting in the waiting room and continues to do so until a bed for the patient becomes available.

Adult patients (patients who are 18 or older) are treated in one of the three pods, A, B, or D. Pods A and B are open 24 hours 7 days a week while Pod D, which mostly functions as a fast-track, opens 9 am every day and closes at 2 am the next day. Pediatric patients can be admitted to the pediatric pod, Pod A, or Pod B depending on the time of day and bed availability in each pod. Pediatric pod is open between the hours of 9 am and 2 am just like Pod D. There are 19 beds in Pod A two of which are trauma beds, which are reserved for ESI1 patients only. There are 16 beds in Pod B, 15 beds in Pod D, and 9 beds in the pediatric pod. Note however that these official numbers do not represent the actual capacities of each pod. For various reasons, the actual cap on the number of patients in each pod can be slightly above or below these numbers. As to which patients get priority over the others and which pod is preferred for each patient, the rules stated in Table EC.1 are used. Note that within each priority level the ordering is according to First-Come-First-Served.

The first stage in the ED (ED workup): We did not explicitly model the interactions among the patients, nurses, and physicians mainly because we did not have any data that would help us formalize such interactions and capture them in a realistic manner within a simulation model. Once a patient is in the ED, the model simply keeps the patient in an ED bed for some random amount of time. Clearly, in reality, how long the patient stays in the bed would depend on staffing levels, i.e., how many nurses and physicians are working during the patient's sojourn in the ED. We capture this effect by allowing the ED workup time to depend on the time-of-day (based on which staffing levels change) along with the patient's ESI level and whether she is an adult or pediatric patient. Tables EC.3 and EC.4 in Appendix E.4 lists the probability distributions that we found to be the best fit to the ED workup time data using the Kolmogorov-Smirnov test.

The second stage in the ED (discharge process or boarding): At the end of the first stage in the ED, a disposition decision for the patient is made. If the decision is to discharge the patient from the ED, the discharge process starts. During this process the patient continues to keep the ED bed occupied. At the end of the discharge process the patient leaves the ED (and the simulation as well) and the ED bed becomes available for patients who are waiting for admission to the ED or future ED patients. The discharge process time depends on the time-of-day, the patient's ESI level, and whether the patient is an adult or pediatric patient. Tables EC.5 and EC.6 in Appendix E.4 list the probability distributions that we found to be the best fit to the data using the Kolmogorov-Smirnov test.

If the disposition decision is to admit the patient to the hospital, the ED requests a bed from the hospital. If there is a bed available, TPP is initiated right away. Otherwise, the request joins a queue which proceeds in a FCFC manner. Thus, in the case of hospital admission, the second stage service corresponds to the patient boarding time, which is equal to any queueing time for a hospital bed plus the TPP time. Unfortunately, we did not have data that would help us estimate the TPP times. Therefore, following Shi et al. (2016), we assumed that TPP has lognormal distribution with mean 3.3 hours and coefficient of variation equal to 0.6. However, we truncated the distribution at 12 hours. We made minor adjustments to these parameters when validating our simulation model as discussed in E.2. As soon as the admitted patient is transferred to the hospital, the patient's hospital length of stay is initiated and the ED bed becomes available for other patients.

Hospital stay: The hospital is modeled as a multi-server queue with homogeneous servers that correspond to hospital beds and arrivals coming from ED admissions only. This is a simplification of the actual system since ED is in fact not the only source of arrivals for the hospital, and hospital beds are in fact not homogeneous because patients can be accepted to different specialty wards depending on their conditions. However, given our focus on the ED, not the hospital, and lack of data needed for a detailed simulation model that can be validated, this modeling approach adequately captures the impact of limited hospital bed availability on ED boarding times.

As soon as a patient is transferred to a hospital bed, the patient's hospital length-of-stay (hLOS) starts. Once the patient's hLOS is over, the patient is immediately discharged from the hospital (and leaves the simulation) and the hospital bed becomes available for new admits. In modeling hLOS, we used the two-time scale approach used by Shi et al. (2016). Specifically, we set *hLOS time* to be equal to *hLOS-day* plus *hospital discharge time*, where hLOS-day is equal to the number of nights the patient stays in the hospital and hospital discharge time is equal to the time between the midnight of the patient's discharge day and discharge time of the patient. Note that a same-day discharge patient would have hLOS-day = 0.

Because we did not have the data to estimate hLOS-day and hospital discharge time, we used the estimates provided by Shi et al. (2016) (with some slight adjustments). Specifically, we assumed that hLOS time depends on whether the patient was admitted to the hospital in the morning or afternoon. Adopting the names used by Shi et al. (2016) and referring to the morning and afternoon patients as ED-AM patients and ED-PM patients, respectively, we assumed that hLOS-day for both ED-AM and ED-PM patients have lognormal distributions with parameters estimated using Table 11 of Shi et al. (2021a). We used the period-1 data because period-2 data correspond to a time period when the hospital the data came from was implementing an early-discharge policy, and we made slight adjustments by enforcing the hLOS-day for ED-PM patients to be at least 1 day and for all patients to be at most 21 days. As a result of this analysis, in our simulations, we assumed that 11.29 percent of the ED-AM patients have hLOS-day = 0 and the remaining ED-AM patients have hLOS-day $= \min(21, \lceil X \rceil)$ where X has lognormal distribution with mean 3.86 and standard deviation 3.93. For ED-PM patients, we assumed that hLOS-day $= \min(21, \lceil Y \rceil)$ where Y has lognormal distribution with mean 4.51 and standard deviation 4.2.

For hospital discharge time distribution (the time at which the patient is discharged on the discharge day), we used the period-1 empirical distribution given in Table 1 of Shi et al. (2021a) with a slight change for patients who are discharged on the same day as their admission. Specifically, for patients whose hospital length-of-stay is at least 1 day, we used the empirical distribution as it is. However, for patients who are discharged on the day of their admission (who can only be ED-AM patients), we assumed that they can only be discharged in the afternoon and used the empirical distribution conditional on the discharge time being in the afternoon. See Table EC.2 for precise specifications of these distributions.

E.2. Model Validation

In order to reach credible conclusions using the simulation model, it is essential that the model is validated. In particular, the simulation model should be producing results that are in line with the data especially for the performance measures based on which our conclusions are formed. In our analysis, we will be primarily using mean ED length-of-stay and the mean number of daily false bed requests and therefore when validating the model, we concentrated on the mean length of stay in the ED over the course of a day. (We cannot use mean number of daily false bed requests for validation because in the current system, no early bed requests are made.)

When validating the simulation model, we picked several parameters, whose values might change from time to time or whose values cannot be determined precisely, as calibration parameters. These were mostly parameters related to the ED and the hospital bed capacity and their choices as calibration parameters were justified due to the following facts: First, in practice, the ED was not

Table EC.2 Probability Distributions for Discharge Times.

discharge time	patients with hLOS$=0$	patients with hLOS≥ 1
0-1	0	0.0015
1-2	0	0.0015
2-3	0	0.0011
3-4	0	0.0009
4-5	0	0.0010
5-6	0	0.0011
6-7	0	0.0015
7-8	0	0.0007
8-9	0	0.0016
9-10	0	0.0132
10-11	0	0.0369
11-12	0	0.0655
12-13	0.1119	0.0977
13-14	0.2221	0.1939
14-15	0.2948	0.2574
15-16	0.1209	0.1056
16-17	0.0696	0.0608
17-18	0.0511	0.0446
18-19	0.0421	0.0368
19-20	0.0371	0.0324
20-21	0.0292	0.0255
21-22	0.0121	0.0106
22-23	0.0054	0.0047
23-24	0.0037	0.0032

operating strictly according to the bed capacities as described in Section E.1. Depending on factors that are impossible to formalize and capture in a simulation model, the capacities in effect stayed below or went above the numbers we reported in Section E.1. Second, even though Pod D was officially open from 9 am to 2 am, in practice the pod typically avoided admitting new patients close to 2 am and appeared to admit very few patients during the first hours of its operation everyday. Third, we did not have data that we could use to directly estimate the mean TPP time. Finally, the true hospital bed capacity for patients who are admitted by the ED depends on many other factors including scheduled surgeries and hospital transfers, which we had to leave out of the simulation model, but also time-of-day as well as day-of-week.

As a result of a series of experiments with different sets of choices for the parameters described above, we were able to validate our simulation model with the following choices: the numbers of beds in Pods A and D respectively were changed from 19 to 12 and from 15 to 17 and the operating hours of Pod D was changed so that it was open from 11 am until 11 pm everyday. The mean TPP time was set to 3.3 hours, and for the hospital bed capacity, we used a schedule where there were 310 beds from 0 am to 10 am and 350 beds for the rest of the day.

Figure EC.1 provides plots of average first stage time (ED workup time), average second stage time (either discharge or boarding time), and average length-of-stay computed over the course of

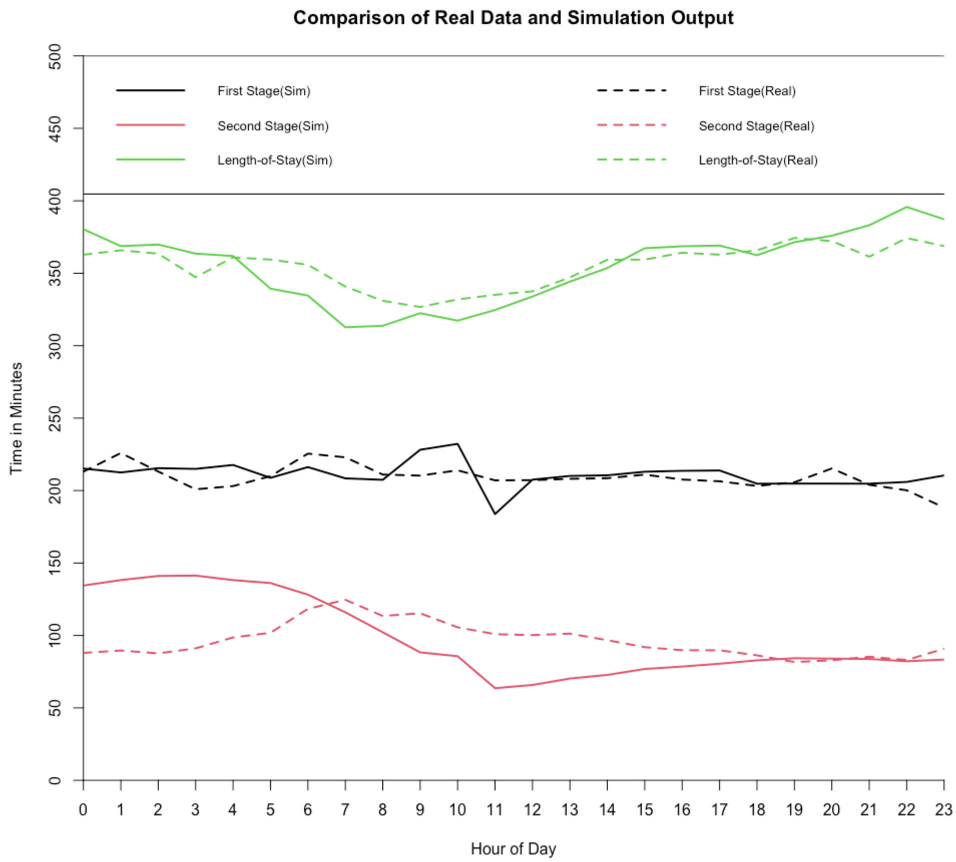

Figure EC.1 Average Length-of-Stay, and First and Second Stage Time Plotted Over the Course of a Day Using UNC ED Data and Simulation Output.

a day using the ED data as well as the simulation output. The simulation results are based on a simulation of length 20 years where the first year data were truncated due to initialization bias. The averages are computed over 24 one-hour time intervals starting from 12:00 am - 12:59 am, which corresponds to time interval 0 in the x-axis, to 11:00 pm - 11:59 pm, which corresponds to time interval 23. The average length-of-stay for a given one-hour period was calculated using length-of-stay times of all patients who arrived at the ED during that time period. The average first stage time for a given one-hour period was calculated using first stage times of all patients whose ED bed assignment was made during that time period. Finally, the average second stage time was calculated using second stage times of all patients for whom disposition decision is made during that time period.

We can see from the figure that the simulation output data provides a good fit to the ED data. In addition, our paired t-test on the hypothesis that the overall length-of-stay from the simulation is the same as the one we obtain from the ED revealed that the hypothesis fails to be rejected at a significance level of 0.05. These suggest that it would be reasonable to use our simulation model as a representation of the ED at least to the extent that the primary focus remains on the average length-of-stay.

E.3. Incorporating BeRT in the Simulation Model

The underlying structure of the simulation model we use to test the performances of policies we propose for making early bed request decisions is essentially the same as the base model described in Section E.1. However, we need to describe how exactly we model bed requests at triage and how we capture the operational impact of the practice in the simulation model.

We assume that the ED uses APT as described in Section 4 and for every patient, at the time of triage, the hospital admission probability for the patient is calculated. In the simulation model, the admission probability for a random patient is modeled as an independent random variable with a probability distribution that depends on the ESI level and age category (adult or pediatric) of the patient, and is estimated using the ED data. These estimated empirical distributions are given in Table EC.15. If the hospital admission probability for the patient is higher than the threshold then a request for a hospital bed is placed for the patient. The threshold level depends on which policy is in place. If CTT is used, then the threshold is set to the right-hand side of (5.9) or (5.10) depending on whether all beds are full or not. If FT is used, the threshold is the policy parameter itself and is set to whatever value the ED and the hospital deemed to be acceptable.

If a decision is made to request a hospital bed at triage, then unlike the case in the base simulation model, at the same time as the patient's start of the ED workup, the bed request is placed. If a hospital bed is available, TPP is initiated; if no beds are available, the request joins a queue. Either way, while TPP is in progress or the request is in a queue, the ED workup for the patient progresses in parallel. If at the completion of the patient's ED workup, the decision is to admit the patient, the patient is either transferred to the main hospital right away, which happens if the TPP for the patient is already complete, or waits for it to complete and then is transferred. If the patient is not admitted to the hospital, then the patient vacates the bed after the discharge process is complete. ED workup times, discharge times, TPP times, and hospital lengths-of-stay are the same as estimated for the base model.

E.4. Estimating Distributions and Parameters for the Simulation Study

As we explained in Section 4, we used data collected during the year 2012. The analysis of these data resulted in the estimates we report in this section. For the fitted distributions, we report the p-value for the Kolmogorov-Smirnov test as well as the mean squared error (MSE). Note that a large value for the p-value indicates a good fit and in cases where we were not able to identify a distribution with a large p-value we picked one with a small MSE.

In what follows, we use the following shortcuts to refer to various probability distributions:

WEIB(β, α): Weibull distribution with scale parameter β and shape parameter α.

ERLA(μ, k): Erlang distribution with k phases and mean phase time μ.

BETA(α_1, α_2): Beta distribution with shape parameters α_1 and α_2.

GAMM(β, α): Gamma distribution with scale parameter β and shape parameter α.

NORM(μ,σ): Normal distribution with mean μ and standard deviation σ.

EXPO(μ): Exponential distribution with mean μ.

MIXEDLOGN($p, \mu_1, \sigma_1, 1-p, \mu_2, \sigma_2$): A mixed distribution that is lognormal with shape and scale parameters respectively μ_1 and σ_1 with probability p and lognormal with shape and scale parameters respectively μ_2 and σ_2 with probability $1-p$.

Tables EC.3 and EC.4 below present the probability distributions fitted for the ED workup time respectively for adult and pediatric patients. The probability distributions depend on the ESI level and time-of-day. In cases where time-of-day does not make a significant difference, a single probability distribution is specified.

Table EC.3 Probability Distributions Fitted for ED Workup Times for Adult Patients.

ESI Level	Time-of-Day	Probability Distribution	p-value	MSE
ESI1	12:00am - 11:59pm	10+WEIB(88.9,.903)	$>.15$	0.00024
ESI2	12:00am - 4:59am	MIXEDLOGN(.97,5.33,.842,.03,2.84,.344)	.381	.03235
ESI2	5:00am - 2:00pm	MIXEDLOGN(.96,5.32,.674,.04,3.27,.495)	.5885	.01729
ESI2	2:00pm - 11:59pm	MIXEDLOGN(.96,5.39,.678,.04,3.39,.574)	.04135	.02434
ESI3	6:00am - 5:59pm	MIXEDLOGN(.88,5.48,.527,.12,4.46,.773)	.04613	.01065
ESI3	6:00pm - 5:59am	MIXEDLOGN(.95,5.37,.592,.05,3.81,.665)	.09353	.01109
ESI4	3:00am - 10:59am	10+GAMM(110,1.22)	.0102	.00138
ESI4	11:00am - 6:59pm	10+GAMM(93.8,1.24)	$<.01$	.00107
ESI4	7:00pm - 2:59am	10+GAMM(106,1.28)	$<.01$	.00216
ESI5	12:00am - 4:59am	11+WEIB(88.9,.885)	$>.15$	.00487
ESI5	5:00am - 11:59pm	10+EXPO(60)	.122	.00139

Table EC.4 Probability Distributions Fitted for ED Workup Times for Pediatric Patients.

ESI Level	Time-of-Day	Probability Distribution	p-value	MSE
ESI1	12:00am - 11:59pm	10+EXPO(77.9)	$>.15$	.00653
ESI2	2:00am - 8:59am	15+EXPO(212)	$>.15$	.00198
ESI2	9:00am - 1:59pm	10+717 × BETA(1.22,2.14)	$>.15$	.00392
ESI2	2:00pm - 1:59am	10+GAMM(163,1.23)	.0598	.00381
ESI3	4:00am - 5:59pm	10+GAMM(87.5,2.16)	$>.15$	.00036
ESI3	6:00pm - 3:59am	10+ERLA(81.6,2)	.112	.00078
ESI4	12:00am - 11:59pm	10+ERLA(55.4,2)	$>.15$	.00026
ESI5	12:00am - 11:59pm	11+ERLA(36.1,2)	$>.15$	.00127

Tables EC.5 and EC.6 below present the probability distributions fitted respectively for the ED discharge time of adult patients and pediatric patients, respectively. The probability distributions depend on the ESI level and time-of-day. In cases where time-of-day does not make a significant difference, a single probability distribution is specified.

Table EC.5 Probability Distributions Fitted for Discharge Times for Adult Patients.

ESI Level	Time-of-Day	Probability Distribution	p-value	MSE
ESI1 and ESI2	3:00am - 7:59am	5+939BETA(.167,2)	$<.01$	.001449
ESI1 and ESI2	8:00am - 2:59am	5+WEIB(43.54,.721)	$<.01$	.008345
ESI3	5:00am - 10:59am	5+WEIB(34.3,.748)	$<.01$	.008397
ESI3	11:00am - 4:59am	5+WEIB(27.1,.755)	$<.01$	.004829
ESI4	2:00am - 8:59am	5+GAMM(39.4,.637)	$<.01$	.008525
ESI4	9:00am - 1:59am	5+EXPO(19.2)	$<.01$	.00148
ESI5	2:00am - 8:59am	5+WEIB(19.9,.789)	.0761	.003894
ESI5	9:00am - 1:59am	5+EXPO(14.4)	$>.15$	.000847

Table EC.6 Probability Distributions Fitted for Discharge Times for Pediatric Patients.

ESI Level	Time-of-Day	Probability Distribution	p-value	MSE
ESI1 and ESI2	9:00am - 1:59pm	5+WEIB(37.1,.7)	$>.15$	.011996
ESI1 and ESI2	2:00pm - 7:59pm	5+EXPO(27.3)	.0588	.017779
ESI1 and ESI2	8:00pm - 8:59am	5+ERLA(33.7,1)	$<.01$	.018725
ESI3	4:00am - 9:59am	5+722 × BETA(.147,3.69)	$<.01$	.006446
ESI3	10:00am - 3:59am	5+EXPO(19.9)	$<.01$	.003133
ESI4 and ESI5	5:00am - 11:59am	5+EXPO(20.1)	$>.15$	.000463
ESI4 and ESI5	12:00pm - 4:59am	5+EXPO(17.5)	$<.01$	.001218

Tables EC.7 through EC.14 list the arrival rates we estimated and used in our simulation analysis. Note that in all these tables, Sun is for Sunday, Mon is for Monday, Tue is for Tuesday, Wed is for Wednesday, Thur is for Thursday, Fri is for Friday, and Sat is for Saturday. One-hour time slots are represented by integers from 0 through 23 with number 0 corresponding to the first time slot of the day from 12:00 am to 12:59 am, number 1 corresponding to the second time slot of the day from 1:00 am to 1:59 am, and the numbering following this pattern until number 23 corresponding to the last time slot that lasts from 11:00 pm to 11:59 pm.

Table EC.7 Expected Number of Arrivals by Hour of Day on Weekdays for Adult Patients Who Are Eventually Admitted to the Hospital.

Hour of Day (Mon)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0.0566	0.2264	0.7358	0.0189	0
1	0.0377	0.1509	0.6038	0.0566	0
2	0.0755	0.2453	0.566	0.0189	0
3	0.0755	0.283	0.6038	0	0
4	0.0189	0.2642	0.5094	0	0
5	0.0189	0.2642	0.4717	0.0189	0
6	0	0.3208	0.4906	0	0
7	0.0377	0.3396	0.6792	0.0377	0
8	0.0189	0.566	0.7358	0.0566	0
9	0.0377	0.717	1.4717	0.0755	0
10	0.0377	0.9623	2.2453	0.0566	0.0189
11	0.0755	1.4528	2.3585	0.0755	0
12	0.1698	1.0943	2.3396	0.0189	0
13	0.0189	0.8302	1.8113	0.0377	0
14	0.0566	1.3208	2.4717	0	0
15	0.1698	1.0377	2.283	0.0755	0
16	0.1132	1.1509	1.8679	0.0566	0
17	0.0189	1.3208	2.2453	0.0377	0
18	0.0377	0.8868	1.5849	0.0189	0
19	0.1321	0.8491	1.8491	0.1321	0.0189
20	0.0943	0.6792	1.434	0.0377	0
21	0.0755	0.5849	1.283	0.0566	0
22	0.0943	0.6792	1.1321	0.0377	0
23	0.1132	0.3208	0.9245	0.0189	0
Hour of Day (Tue - Fri)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0.0481	0.3942	0.6346	0.024	0
1	0.0625	0.2019	0.5433	0.0096	0
2	0.0385	0.2981	0.4615	0.0192	0
3	0.0433	0.1538	0.4471	0.0048	0
4	0.0337	0.226	0.4567	0.0048	0
5	0.0192	0.1442	0.3942	0.0096	0
6	0.0144	0.1971	0.5192	0.0144	0
7	0.0385	0.1971	0.5769	0.0385	0.0048
8	0.0577	0.3798	0.9327	0.0625	0
9	0.0673	0.4952	1.4327	0.0385	0.0048
10	0.0577	0.7115	1.8173	0.0577	0
11	0.0962	0.976	2.1827	0.0673	0
12	0.1154	0.8894	2.2115	0.0288	0.0048
13	0.0769	1.024	2.0865	0.0529	0
14	0.1346	1.1058	2.0144	0.0529	0.0048
15	0.1202	1.2163	2.3029	0.0433	0.0048
16	0.0817	1.024	2.0913	0.0385	0.0048
17	0.0769	0.9327	2.1538	0.0481	0
18	0.101	0.851	1.649	0.0577	0
19	0.0865	0.8606	1.5962	0.0577	0
20	0.0962	0.5721	1.4183	0.0385	0
21	0.0673	0.7212	1.226	0.024	0
22	0.0433	0.5721	1.1154	0.0192	0
23	0.0577	0.524	0.9279	0.0096	0

Table EC.8 Expected Number of Arrivals by Hour of Day on Weekends for Adult Patients Who Are Eventually Admitted to the Hospital.

Hour of Day (Sat & Sun)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0.0381	0.4	0.6286	0.0286	0
1	0.0571	0.3714	0.6	0.0381	0
2	0.0762	0.2857	0.5238	0.019	0
3	0.0381	0.2857	0.3524	0.019	0
4	0.0286	0.2095	0.5429	0	0
5	0.0667	0.1714	0.4286	0	0
6	0.0476	0.2476	0.5333	0.0095	0
7	0.0286	0.2571	0.7143	0.0476	0
8	0.0667	0.3333	0.9619	0.019	0
9	0.0286	0.5905	1.2381	0.1238	0
10	0.0571	0.619	1.5238	0.1333	0
11	0.0476	0.5048	1.4762	0.0857	0
12	0.0381	0.7238	1.5524	0.0381	0.0095
13	0.0286	0.6952	1.7333	0.0571	0
14	0.0762	0.6095	1.781	0.0381	0
15	0.0952	0.6476	1.9238	0.0667	0.0095
16	0.0952	0.6667	1.9619	0.0381	0
17	0.0571	0.5714	1.781	0.0095	0
18	0.019	0.7714	1.5524	0.0095	0
19	0.0762	0.6286	1.3905	0.0286	0
20	0.0476	0.7619	1.1714	0.0381	0
21	0.0667	0.4857	1.181	0.0952	0
22	0.0571	0.5619	1	0.0095	0
23	0.0667	0.5238	0.819	0.0095	0

Table EC.9 Expected Number of Arrivals by Hour of Day on Weekdays for Adult Patients Who Are Eventually Discharged from the ED.

Hour of Day (Mon)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0	0.1132	1.434	0.6415	0.0943
1	0	0.2075	1.0755	0.4717	0.0377
2	0	0.0755	0.7547	0.4151	0.0377
3	0	0.0755	0.7547	0.3208	0.0377
4	0	0	0.6415	0.3208	0.0189
5	0.0189	0.0566	0.8113	0.3585	0.0189
6	0	0.0755	1.0755	0.566	0.0943
7	0	0.0943	1.3396	1.1132	0.1698
8	0	0.1509	2.7925	1.6038	0.434
9	0.0189	0.2075	3.8868	2.4717	0.4528
10	0.0189	0.3774	4.6792	2.7547	0.8679
11	0	0.3585	4.6415	2.3208	0.6792
12	0	0.3774	4.9245	2.6981	0.3396
13	0.0189	0.4528	4.566	2	0.3396
14	0.0189	0.3208	3.9434	1.8868	0.3585
15	0.0189	0.4528	3.3396	1.9623	0.3585
16	0	0.3774	3.4717	1.6792	0.2642
17	0.0189	0.3396	3.6604	1.6792	0.2075
18	0.0189	0.3208	2.9245	1.6415	0.283
19	0	0.6604	3.2642	1.4151	0.1132
20	0.0189	0.283	2.7358	1.566	0.1509
21	0	0.2075	2.283	1.2264	0.1321
22	0	0.283	1.9623	0.9811	0.0566
23	0	0.2075	1.7358	0.8868	0.0566
Hour of Day (Tue - Fri)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0	0.1442	1.2548	0.5577	0.0865
1	0.0048	0.1154	1.0577	0.5096	0.0962
2	0.0096	0.1058	0.9087	0.4663	0.0721
3	0.0048	0.1106	0.8269	0.3029	0.0673
4	0.0048	0.0577	0.6731	0.2692	0.0288
5	0.0048	0.0625	0.7356	0.3221	0.0577
6	0	0.0721	0.8462	0.4183	0.0913
7	0.0048	0.0865	1.2981	0.7837	0.1971
8	0.0192	0.2067	2.1971	1.4231	0.375
9	0.0144	0.1779	3.3702	2.3317	0.5769
10	0.0144	0.226	3.8462	2.2788	0.4808
11	0.0144	0.3413	3.8029	2.0769	0.5769
12	0.0048	0.3173	3.8846	1.9519	0.4471
13	0.0048	0.3846	3.6635	1.7067	0.3846
14	0.0144	0.3413	3.5288	1.6346	0.2933
15	0.0288	0.4087	3.4808	1.4615	0.2644
16	0.0096	0.3798	3.3077	1.6779	0.2788
17	0.024	0.3702	3.0865	1.5577	0.2837
18	0	0.3125	3.0865	1.5865	0.2115
19	0.0144	0.2885	3.1154	1.7356	0.25
20	0.0048	0.2981	2.7981	1.1923	0.1683
21	0.0048	0.3654	2.6635	1.2115	0.1683
22	0	0.2692	2	0.899	0.1875
23	0.0096	0.274	1.6875	0.7933	0.0769

Table EC.10 Expected Number of Arrivals by Hour of Day on Weekends for Adult Patients Who Are Eventually Discharged from the ED.

Hour of Day (Sat & Sun)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0.0095	0.2571	1.7238	0.7714	0.0762
1	0	0.181	1.4286	0.6381	0.0571
2	0	0.1714	1.1619	0.7619	0.0381
3	0	0.1619	1.1619	0.7333	0.0571
4	0	0.1048	0.8762	0.4667	0.0762
5	0	0.1143	0.9143	0.3619	0.0286
6	0	0.0952	0.9143	0.4952	0.0286
7	0	0.0667	1.2286	0.6286	0.1048
8	0	0.1048	1.8286	1.3524	0.2571
9	0.019	0.1714	2.1238	2.1048	0.3619
10	0	0.1619	2.8476	2.3048	0.3429
11	0	0.1619	3.2381	2.419	0.3619
12	0	0.2095	3.3238	2.2857	0.3143
13	0.019	0.2762	3.5333	2.2857	0.3619
14	0	0.3048	3.219	1.981	0.4
15	0	0.2571	3.2286	1.981	0.3429
16	0	0.2667	3.1714	1.7905	0.3143
17	0.0286	0.3238	3.3714	1.5619	0.2381
18	0.0286	0.3143	2.7905	1.619	0.2286
19	0	0.3714	2.7524	1.7619	0.219
20	0	0.2762	2.6476	1.419	0.2476
21	0.0095	0.3143	2.6571	1.4095	0.181
22	0.0095	0.2857	2.0571	1.0762	0.181
23	0	0.1619	1.9333	0.8857	0.0857

Table EC.11 Expected Number of Arrivals by Hour of Day on Weekdays for Pediatric Patients Who Are Eventually Admitted to the Main Hospital.

Hour of Day (Mon)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0	0.0566	0.1321	0	0.0189
1	0	0.0566	0.0943	0	0
2	0.0189	0.0189	0.0755	0	0
3	0	0.0189	0.0566	0	0
4	0	0.0189	0.0377	0	0
5	0	0.0189	0.0377	0.0377	0
6	0	0.0189	0.0189	0.0189	0
7	0	0.0189	0.0377	0	0
8	0.0189	0	0.1509	0	0.0189
9	0	0.0943	0.0566	0	0
10	0	0.0943	0.1509	0	0
11	0.0189	0.1698	0.2453	0.0189	0
12	0.0189	0.2264	0.2075	0.0189	0
13	0.0189	0.1887	0.2453	0.0189	0
14	0	0.1321	0.1887	0.0189	0
15	0.0189	0.1132	0.1887	0	0
16	0.0189	0.2075	0.2264	0	0
17	0	0.1321	0.2453	0.0189	0
18	0	0.2453	0.2642	0.0566	0
19	0	0.1132	0.1887	0.0189	0
20	0.0189	0.1698	0.3396	0.0566	0
21	0	0.1887	0.3208	0	0
22	0	0.0566	0.1509	0.0189	0
23	0	0.0377	0.0377	0	0
Hour of Day (Tue - Fri)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0	0.0865	0.0962	0.0048	0
1	0.0048	0.0577	0.0817	0	0
2	0	0.0144	0.0673	0.0096	0
3	0	0.0673	0.0529	0.0096	0
4	0.0048	0.024	0.0433	0.0048	0
5	0.0048	0.0288	0.0288	0.0048	0
6	0.0048	0.0192	0.0288	0	0
7	0.0096	0.0433	0.0337	0.0048	0
8	0.0048	0.0288	0.0577	0.0144	0
9	0.0144	0.0481	0.1106	0.024	0
10	0.0144	0.0817	0.1442	0.0048	0
11	0.0192	0.0865	0.2404	0.0048	0
12	0	0.1346	0.1827	0.0144	0.0048
13	0.0048	0.0865	0.1683	0.0144	0
14	0	0.1202	0.1827	0.0096	0
15	0	0.1875	0.2067	0.0048	0.0048
16	0.0144	0.1635	0.1875	0	0
17	0.0144	0.1875	0.2837	0.024	0
18	0.0288	0.1635	0.2692	0.0048	0
19	0.0144	0.1635	0.2212	0.0385	0
20	0.0288	0.1298	0.2019	0.024	0
21	0.0096	0.1202	0.1346	0.0288	0
22	0.0096	0.1538	0.1635	0.0144	0
23	0	0.0913	0.2067	0.0096	0

Table EC.12 Expected Number of Arrivals by Hour of Day on Weekends for Pediatric Patients Who Are Eventually Admitted to the Hospital.

Hour of Day (Sat & Sun)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0.0095	0.0857	0.1524	0.0286	0
1	0	0.0857	0.0667	0.0095	0
2	0	0.0571	0.0667	0.0095	0
3	0	0.0857	0.0286	0	0
4	0	0.019	0.0476	0.0095	0
5	0	0.0095	0.019	0	0
6	0	0.019	0.0095	0	0
7	0	0.0286	0.019	0	0
8	0	0.019	0.0286	0	0
9	0.0095	0.0571	0.1048	0.0286	0
10	0.019	0.0571	0.0476	0.019	0
11	0	0.1619	0.2286	0	0
12	0.0095	0.0952	0.2286	0.0286	0
13	0	0.0857	0.1619	0.019	0
14	0.019	0.1143	0.1524	0.0286	0
15	0.019	0.1048	0.2476	0.0381	0
16	0	0.0857	0.2286	0.0095	0
17	0	0.1143	0.1619	0.0095	0
18	0	0.1048	0.1714	0.0286	0
19	0.0286	0.1048	0.1905	0.0286	0
20	0	0.0762	0.1905	0.0095	0
21	0	0.0857	0.1429	0.0286	0
22	0	0.1524	0.1238	0.0095	0
23	0.019	0.0762	0.1238	0.0095	0

Table EC.13 Expected Number of Arrivals by Hour of Day on Weekdays for Pediatric Patients Who Are Eventually Discharged From the ED.

Hour of Day (Mon)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0	0.0755	0.2642	0.5094	0.0189
1	0	0.0377	0.3585	0.2642	0.0189
2	0	0.0377	0.1321	0.1321	0.0189
3	0	0	0.2075	0.2075	0.0189
4	0	0	0.0943	0.1887	0
5	0	0	0.1321	0.0943	0
6	0	0.0189	0.0943	0.1132	0.0189
7	0	0.0377	0.2264	0.2075	0.0943
8	0	0	0.4151	0.2642	0.0377
9	0	0.0566	0.2453	0.434	0.1132
10	0	0.0377	0.5283	0.566	0.1132
11	0	0.0755	0.6038	0.6226	0.0943
12	0	0.0189	0.5094	0.5849	0.2075
13	0	0.1132	0.6981	0.4151	0.1132
14	0.0189	0.0755	0.5849	0.5094	0.1321
15	0	0.1698	0.5094	0.6415	0.0943
16	0	0.1132	0.7736	0.6792	0.0566
17	0	0.1509	1.0377	0.8679	0.1132
18	0	0.2453	0.9623	1.0566	0.2642
19	0	0.1509	0.6792	0.9434	0.0755
20	0	0.1321	0.8868	1.1132	0.2642
21	0	0.1132	0.6604	0.9623	0.0943
22	0	0.0943	0.5472	0.7547	0.1321
23	0	0.0377	0.3208	0.5094	0.1132
Hour of Day (Tue - Fri)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0	0.0481	0.2404	0.274	0.0769
1	0	0.0337	0.1683	0.2452	0.0288
2	0	0.0048	0.1058	0.2356	0.0144
3	0	0.0096	0.0673	0.1298	0.0096
4	0	0.0192	0.0865	0.1731	0.0337
5	0	0.0144	0.0865	0.1779	0.0096
6	0	0.0096	0.0625	0.125	0.0192
7	0	0	0.1635	0.1683	0.0337
8	0	0.0096	0.3125	0.274	0.0481
9	0	0.0433	0.4519	0.3558	0.149
10	0	0.0529	0.4904	0.4519	0.149
11	0	0.1298	0.4856	0.5144	0.1394
12	0	0.0721	0.5865	0.4663	0.1058
13	0	0.101	0.5962	0.4375	0.1058
14	0	0.1154	0.6106	0.4904	0.0721
15	0	0.1779	0.4183	0.5385	0.0529
16	0	0.0865	0.5481	0.476	0.0673
17	0	0.1538	0.7933	0.75	0.1298
18	0	0.1442	0.8317	0.8942	0.1538
19	0	0.1058	0.8269	1.0433	0.1298
20	0	0.1058	0.6875	0.9471	0.1923
21	0	0.125	0.6923	0.8077	0.1683
22	0	0.0721	0.6298	0.7115	0.0865
23	0	0.0385	0.3173	0.4856	0.0673

Table EC.14 Expected Number of Arrivals by Hour of Day on Weekends for Pediatric Patients Who Are Eventually Discharged From the ED.

Hour of Day (Sat & Sun)	ESI1	ESI2	ESI3	ESI4	ESI5
0	0	0.0476	0.2857	0.3905	0.0857
1	0	0.0381	0.3238	0.2667	0.0476
2	0	0	0.1143	0.2667	0.019
3	0	0.0381	0.0952	0.2	0.019
4	0	0.019	0.1238	0.1048	0.019
5	0	0.0095	0.1048	0.1714	0
6	0	0.0381	0.1333	0.181	0.0286
7	0	0.0095	0.1333	0.2762	0.0286
8	0	0.0286	0.1524	0.3429	0.0762
9	0	0.019	0.2952	0.7714	0.1048
10	0	0.019	0.3714	0.9143	0.1143
11	0	0.0476	0.4667	0.9524	0.2286
12	0	0.0667	0.6857	0.9429	0.1619
13	0	0.0571	0.6476	0.9143	0.1524
14	0	0.1238	0.5429	0.781	0.2095
15	0	0.0952	0.6667	0.8381	0.1143
16	0	0.1238	0.4952	0.9619	0.1333
17	0	0.0857	0.781	0.8762	0.0952
18	0.019	0.0476	0.581	0.7714	0.0952
19	0	0.1143	0.619	1.2	0.1048
20	0	0.0571	0.5619	1.1333	0.181
21	0	0.1238	0.781	0.8286	0.1429
22	0	0.0857	0.6	0.7429	0.0952
23	0	0.0857	0.5429	0.5429	0.0952

The probability of hospital admission for a random patient is a random variable and we estimated its probability distribution conditional on the patient age category (adult or pediatric) and ESI level. The estimated empirical distributions are given in Table EC.15. Note that in the table EMP$(p_1, v_1, p_2, v_2, \dots, p_K, v_K)$ denotes a discrete random variable Z with $P\{Z \in (v_1, v_2, \dots, v_K)\} = 1$, $P\{Z = v_1\} = p_1$, and $P\{Z \leq v_k = p_k\}$ for $1 \leq k \leq K$. In other words, p_i is the cumulative probability corresponding to v_i.

Table EC.15 Probability Distributions for Hospital Admission Probabilities.

ESI Level	Age Group	Probability Distribution
ESI1	Adult	EMP(0.002,0.848,0.264,0.901,0.49,0.939,0.729,0.962,0.986,0.977,0.994,0.992,1,1)
ESI2	Adult	EMP(0.001,0.528,0.308,0.661,0.554,0.764,0.799,0.838,0.999,0.985,1,1)
ESI3	Adult	EMP(0.001,0.175,0.388,0.275,0.648,0.425,0.848,0.525,0.999,0.875,1,1)
ESI4	Adult	(0.003,0.018,0.56,0.03,0.841,0.055,0.961,0.083,1,0.1)
ESI5	Adult	EMP(0.001,0.003,0.609,0.006,0.879,0.01,0.982,0.016,1,0.02)
ESI1	Pediatric	EMP(0.001,0.854,0.956,0.923,1,0.95)
ESI2	Pediatric	EMP(0.002,0.548,0.323,0.559,0.933,0.741,0.999,0.855,1,0.9)
ESI3	Pediatric	EMP(0.003,0.204,0.952,0.387,0.999,0.679,1,0.77)
ESI4	Pediatric	EMP(0.002,0.018,0.969,0.045,1,0.18)
ESI5	Pediatric	EMP(0.001,0.004,0.961,0.009,1,0.01)

E.5. Additional Plots Based on the Simulation Study for Scenario 1

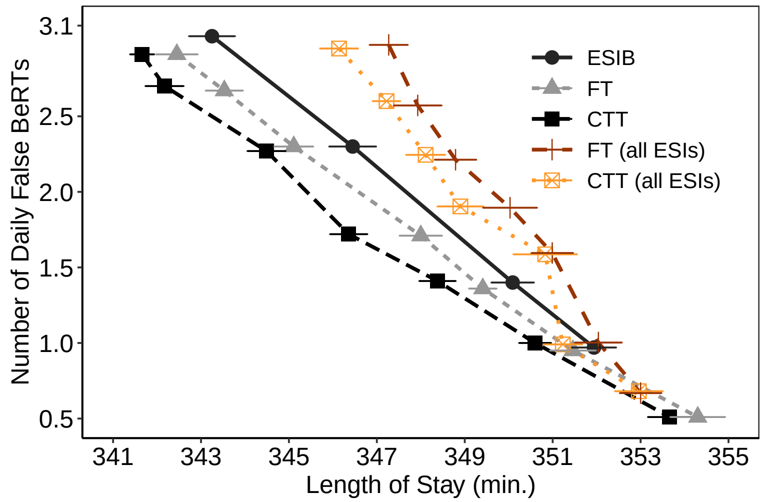

Figure EC.2 Long-Run Average Length-of-Stay for All Patients and Long-Run Average Number of Daily False Bed Requests under ESIB, FT, CTT, FT applied to all ESI levels, and CTT applied to all ESI levels for Scenario 1.

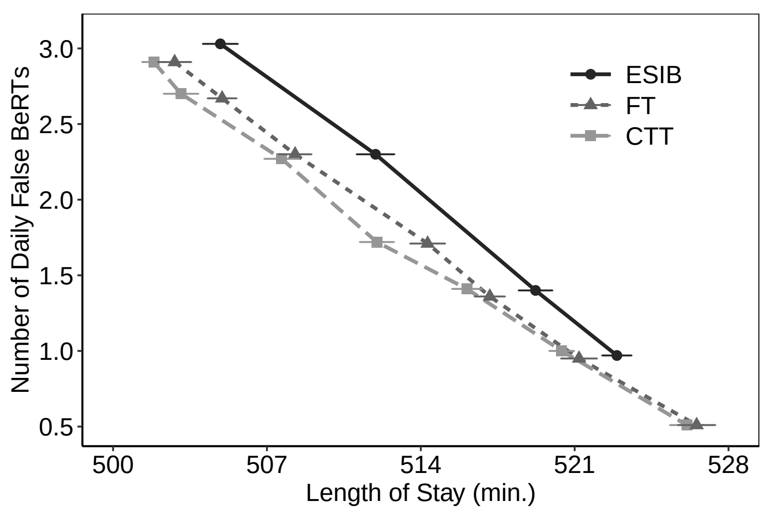

Figure EC.3 Long-Run Average Length-of-Stay for Admitted Patients and Long-Run Average Number of Daily False Bed Requests under ESIB, FT, and CTT for Scenario 1.

E.6. Additional Plots Based on the Simulation Study for Scenario 2

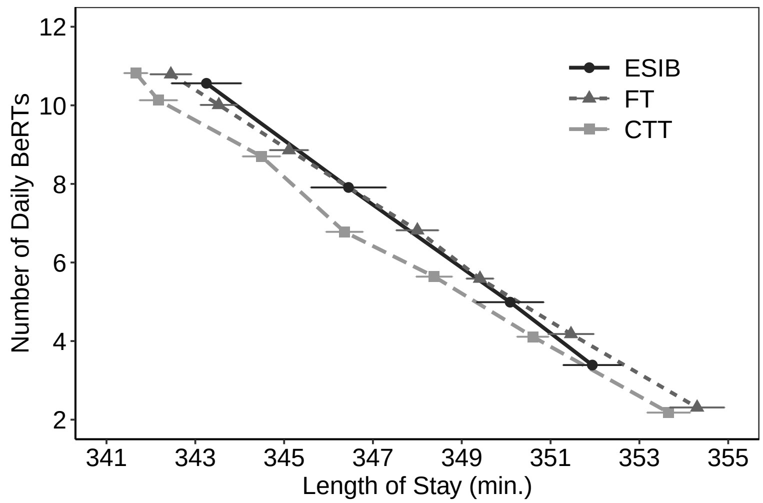

Figure EC.4 Long-Run Average Length-of-Stay and Long-Run Average Number of Total Daily Early Bed Requests under ESIB, FT, and CTT for Scenario 1.

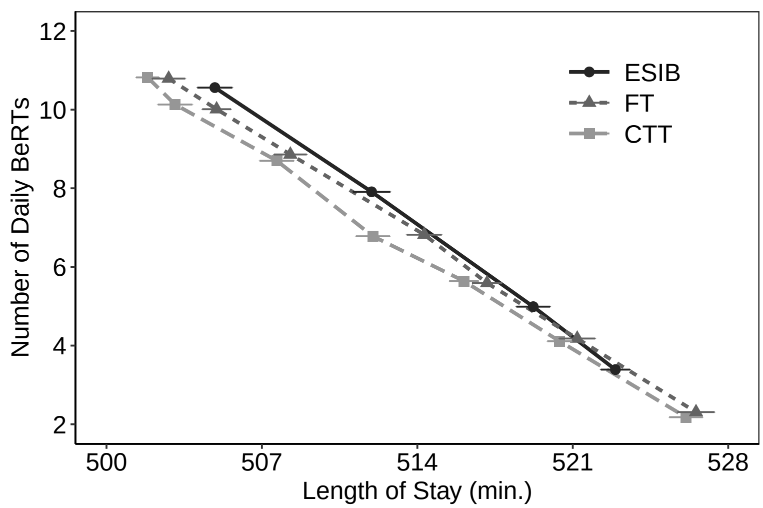

Figure EC.5 Long-Run Average Length-of-Stay for Admitted Patients and Long-Run Average Number of Total Daily Early Bed Requests under ESIB, FT, and CTT for Scenario 1.

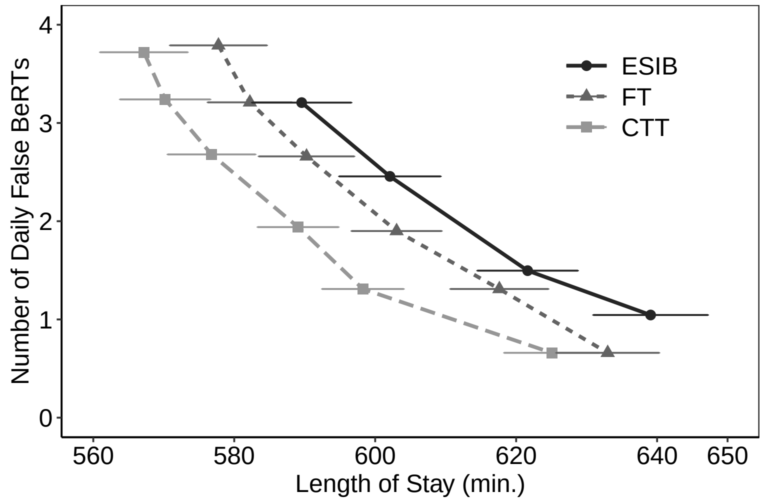

Figure EC.6 Long-Run Average Length-of-Stay for Admitted Patients and Long-Run Average Number of Daily False Bed Requests under ESIB, FT, and CTT for Scenario 2 with 400 beds.

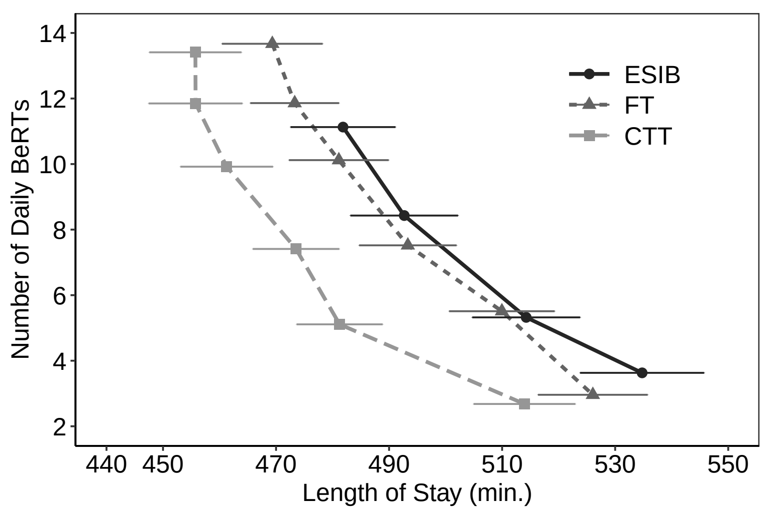

Figure EC.7 Long-Run Average Length-of-Stay and Long-Run Average Number of Total Daily Early Bed Requests under ESIB, FT, and CTT for Scenario 2 with 400 beds.

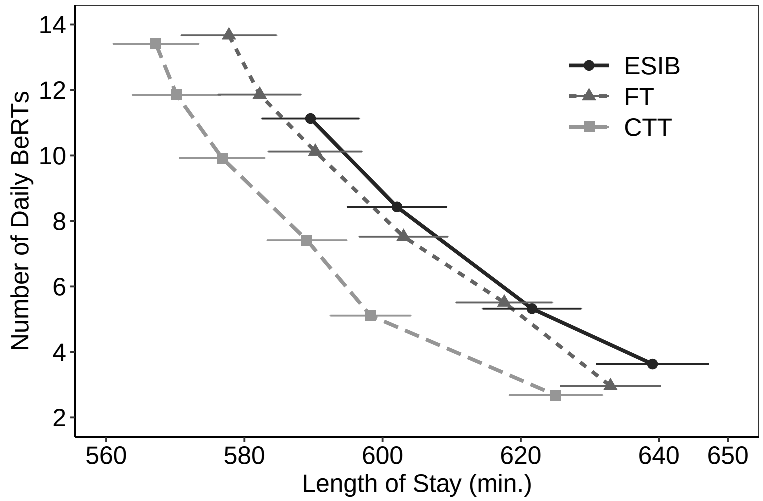

Figure EC.8 Long-Run Average Length-of-Stay for Admitted Patients and Long-Run Average Number of Total Daily Early Bed Requests under ESIB, FT, and CTT for Scenario 2 with 400 beds.

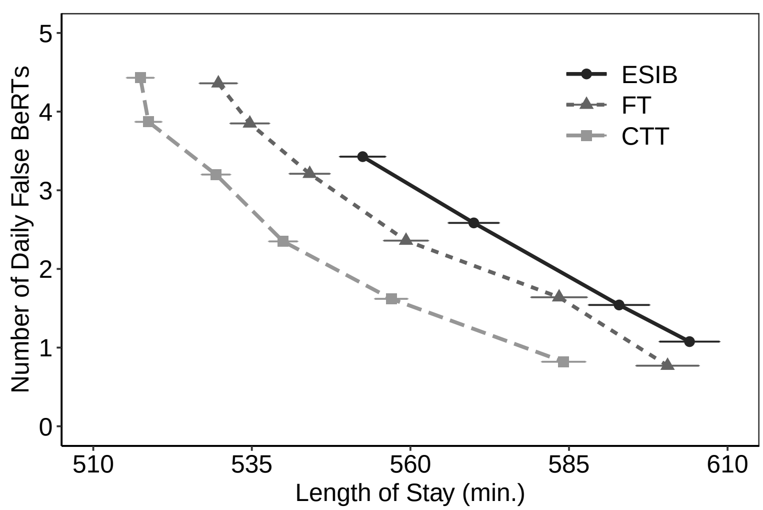

Figure EC.9 Long-Run Average Length-of-Stay for Admitted Patients and Long-Run Average Number of Daily False Bed Requests under ESIB, FT, and CTT for Scenario 2 with 450 beds.

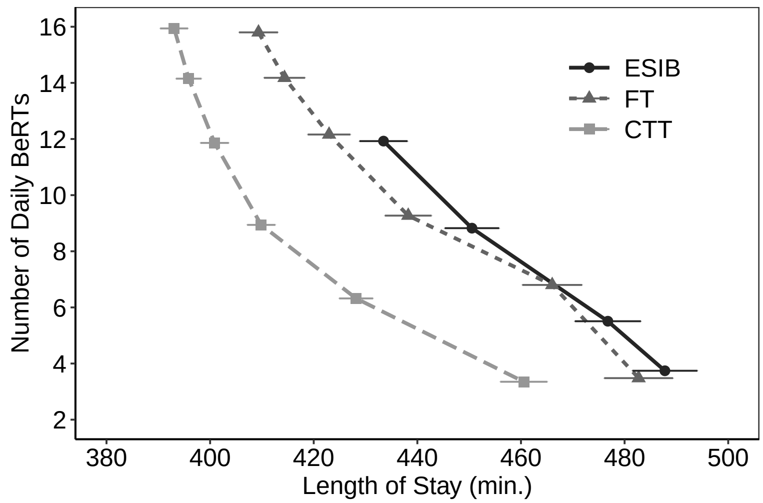

Figure EC.10 Long-Run Average Length-of-Stay and Long-Run Average Number of Total Daily Early Bed Requests under ESIB, FT, and CTT for Scenario 2 with 450 beds.

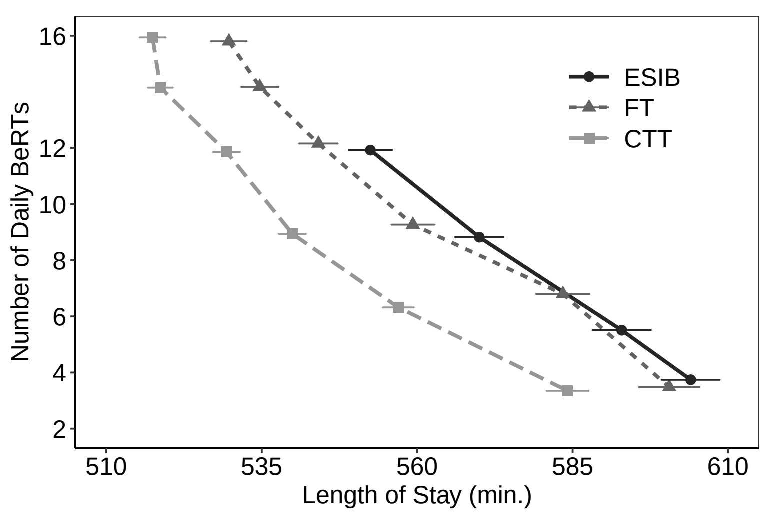

Figure EC.11 Long-Run Average Length-of-Stay for Admitted Patients and Long-Run Average Number of Total Daily Early Bed Requests under ESIB, FT, and CTT for Scenario 2 with 450 beds.